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ON THE SPECTRUM OF A FAMILY OF PRECONDITIONED BLOCK TOEPLITZ MATRICES*

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Abstract. Research on preconditioning Toeplitz matrices with circulant matrices has been active recently. The preconditioning technique can be easily generalized to block Toeplitz matrices. That is, for a block Toeplitz matrix T consisting of $N \times N$ blocks with $M \times M$ elements per block, a block circulant matrix R is used with the same block structure as its preconditioner. In this research, the spectral clustering property of the preconditioned matrix $R^{-1}T$ with T generated by two-dimensional rational functions $T(z_x, z_y)$ of order (p_x, q_x, p_y, q_y) is examined. It is shown that the eigenvalues of $R^{-1}T$ are clustered around unity except at most $O(M\gamma_y + N\gamma_x)$ outliers, where $\gamma_x = \max(p_x, q_x)$ and $\gamma_y = \max(p_y, q_y)$. Furthermore, if T is separable, the outliers are clustered together such that $R^{-1}T$ has at most $(2\gamma_x+1)(2\gamma_y+1)$ asymptotic distinct eigenvalues. The superior convergence behavior of the preconditioned conjugate gradient (PCG) method over the conjugate gradient (CG) method is explained by a smaller condition number and a better clustering property of the spectrum of the preconditioned matrix $R^{-1}T$.

Key words. block Toeplitz matrix, preconditioned conjugate gradient method

AMS(MOS) subject classifications. 65F10, 65F15

1. Introduction.

The systems of linear equations associated with block Toeplitz matrices arise in many two-dimensional digital signal processing applications, such as linear prediction and estimation [9], [12], [13], image restoration [7], and the discretization of constant-coefficient partial differential equations. To solve the block Toeplitz system $T\mathbf{u} = \mathbf{b}$, where T is an $N \times N$ matrix with $M \times M$ blocks, by direct methods, such as Levinson-type algorithms, requires $O(M^3N^2)$ operations [2], [14], [17]. Recently, there has been active research on the application of iterative methods such as the preconditioned conjugate gradient (PCG) method to the solution of Toeplitz systems. To accelerate the convergence rate, various preconditioners have been proposed for symmetric positive definite (SPD) Toeplitz matrices [6], [8], [10], [15]. The proposed preconditioning techniques can be easily generalized to block Toeplitz matrices. Simply speaking, we construct the preconditioner with a block circulant matrix R that has the same block structure as T. Since both $R^{-1}\mathbf{w}$ and $T\mathbf{w}$, where $\mathbf{w}$ denotes an arbitrary vector of length MN, can be performed with $O(MN \log MN)$ operations via two-dimensional fast Fourier transform, the computational complexity per PCG iteration is $O(MN \log MN)$ only. The PCG method can be much more attractive than direct methods for solving block Toeplitz systems if it converges fast.

The convergence rate of the PCG method depends on the eigenvalue distribution of the preconditioned matrix $R^{-1}T$ [1]. Generally speaking, the PCG method converges faster if $R^{-1}T$ has clustered eigenvalues and/or a small condition number. The spectral properties of preconditioned point Toeplitz matrices have been extensively studied. Chan and Strang [3], [5] have proved that, for a Toeplitz matrix with a positive generating function in the Wiener class, the spectrum of the preconditioned matrix has eigenvalues clustered around unity except for a finite number of outliers. If the Toeplitz matrix is generated by a positive *rational* function

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$A(z)/B(z) + A(z^{-1})/B(z^{-1})$ in the Wiener class, an even stronger result has been derived by Trefethen [16] and Ku and Kuo [11]. That is, if $A(z)$ and $B(z)$ are polynominals in z of orders p and q without common roots, the number of outliers is equal to $2\max(p,q)$ and the PCG method converges in at most $\max(p,q)+1$ iterations asymptotically (see [11] and the discussion in §4 below). Therefore, an $N \times N$ preconditioned rational Toeplitz system can be solved with $\max(p,q) \times O(N \log N)$ operations.

The spectral properties of preconditioned block Toeplitz matrices have not yet been very well understood. In this research, we analyze the spectral *clustering* property for a class of preconditioned block Toeplitz matrices. The block Toeplitz matrix under consideration has a two-dimensional quadrantally-symmetric generating sequence generated by a rational function in the Wiener class (see the definition in §3). We divide our discussion into two cases depending on whether or not the generating sequence is separable. When the block Toeplitz matrix has a separable generating sequence, the spectrum of the preconditioned block Toeplitz can be easily derived by using the preconditioned point Toeplitz result as given in [11]. However, we derive the preconditioned point Toeplitz result from a new viewpoint in this paper so that the same approach can be used for both separable and nonseparable cases. With this viewpoint, we interpret the operation $T\mathbf{w}$, where T is an $MN \times MN$ block Toeplitz matrix, as a two-dimensional constant-coefficient mask operating on a certain two-dimensional sequence construction based on $\mathbf{w}$.

Our main results can be summarized as follows. Let T be an $MN \times MN$ doubly symmetric block Toeplitz matrix generated by a rational function of order (p_x, q_x, p_y, q_y), $\gamma_x = \max(p_x, q_x)$ and $\gamma_y = \max(p_y, q_y)$. For the separable generating sequence case, the eigenvalues of $R^{-1}T$ are clustered together such that it has asymptotically $(2\gamma_x+1)(2\gamma_y+1)$ distinct eigenvalues. The PCG method converges asymptotically in at most $\gamma_x\gamma_y+1$ iterations, and the complexity of the PCG method is therefore $O(MN \log MN)$. For the nonseparable generating sequence case, the eigenvalues of $R^{-1}T$ are clustered around unity except for at most $O(M\gamma_y + N\gamma_x)$ outliers. Since the number of outliers is proportional to M and N, rather than being $O(1)$ as in the point Toeplitz case, the convergence rate of the PCG method cannot be completely characterized by the number of outliers. The condition number $\kappa(R^{-1}T)$ should also be taken into account. For this case, the superior performance of the PCG method over the CG method is explained by a better spectral clustering property as well as a smaller condition number of the preconditioned matrix $R^{-1}T$.

The outline of this paper is as follows. The construction of the block circulant preconditioner R for block Toeplitz matrices T is presented in §2. In §3, we study the spectral clustering property of preconditioned block Toeplitz matrices $R^{-1}T$. Toeplitz matrices with separable and nonseparable generating sequences are examined, respectively, in §§3.1 and 3.2. Numerical results are given in §4 to assess the performance of the PCG method.

2. Construction of block Toeplitz preconditioners. Let T be a block Toeplitz matrix consisting of $N \times N$ blocks with $M \times M$ elements per block, which can be expressed as

$$T = \begin{bmatrix} T_0 & T_{-1} & \cdot & T_{2-N} & T_{1-N} \\ T_1 & T_0 & T_{-1} & \cdot & T_{2-N} \\ \cdot & T_1 & T_0 & \cdot & \cdot \\ T_{N-2} & \cdot & \cdot & \cdot & T_{-1} \\ T_{N-1} & T_{N-2} & \cdot & T_1 & T_0 \end{bmatrix}, \tag{2.1}$$

where T_n with $|n| \leq N-1$ are $M \times M$ Toeplitz matrices with elements

$$[T_n]_{i,j} = t_{i-j,n} \quad \text{where} \quad 1 \leq i, \quad j \leq M.$$

Note that T is also known as the doubly Toeplitz matrix. The $MN \times MN$ block Toeplitz matrix T is completely characterized by the two-dimensional sequence

$$t_{m,n} \quad \text{where} \quad |m| \leq M-1, \ |n| \leq N-1, \tag{2.2}$$

known as the generating sequence of T. To construct the preconditioner for T, we generalize the idea in [5], [10], and [15], and consider an $MN \times MN$ block circulant matrix of the form

$$R = \begin{bmatrix} R_0 & R_{N-1} & \cdot & R_2 & R_1 \\ R_1 & R_0 & R_{N-1} & \cdot & R_2 \\ \cdot & R_1 & R_0 & \cdot & \cdot \\ R_{N-2} & \cdot & \cdot & \cdot & R_{N-1} \\ R_{N-1} & R_{N-2} & \cdot & R_1 & R_0 \end{bmatrix}, \tag{2.3}$$

where R_n with $0 \leq n \leq N-1$ are $M \times M$ circulant matrices with elements

$$[R_n]_{i,j} = r_{(i-j) \bmod M,n} \quad \text{where} \quad 1 \leq i,j \leq M.$$

Thus, the block circulant matrix R is completely characterized by the two-dimensional sequence

$$r_{m,n} \quad \text{where} \quad 0 \leq m \leq M-1, \ 0 \leq n \leq N-1. \tag{2.4}$$

The construction of R based on T is described below.

In (2.1) and (2.3), linear operators T and R are expressed in matrix form. Block Toeplitz (or circulant) systems are in fact just one way to describe linear (or circular) convolutions between two two-dimensional sequences. In the current context, it is more convenient to characterize T (or R) in terms of the relationship between input and output vectors. Consider two arbitrary MN-dimensional vectors $\mathbf{w}$ and $\mathbf{v}$ related via $\mathbf{v} = T\mathbf{w}$. By using the natural rowwise ordering, we can rearrange elements of these vectors into two-dimensional sequences

$$w_{m,n} \text{ and } v_{m,n} \quad \text{where} \quad 0 \leq m \leq M-1, \ 0 \leq n \leq N-1. \tag{2.5}$$

Then, the block Toeplitz system $\mathbf{v} = T\mathbf{w}$ can be interpreted as a linear operator characterized by the two-dimensional mask

(2.6)
$$\begin{bmatrix} t_{M-1,-N+1} & t_{M-2,-N+1} & \cdots & t_{0,-N+1} & \cdots & t_{-M+2,-N+1} & t_{-M+1,-N+1} \\ t_{M-1,-N+2} & t_{M-2,-N+2} & \cdots & t_{0,-N+2} & \cdots & t_{-M+2,-N+2} & t_{-M+1,-N+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,0} & t_{M-2,0} & \cdots & t_{0,0} & \cdots & t_{-M+2,0} & t_{-M+1,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,N-2} & t_{M-2,N-2} & \cdots & t_{0,N-2} & \cdots & t_{-M+2,N-2} & t_{-M+1,N-2} \\ t_{M-1,N-1} & t_{M-2,N-1} & \cdots & t_{0,N-1} & \cdots & t_{-M+2,N-1} & t_{-M+1,N-1} \end{bmatrix}$$

operating on an extended sequence

$$\bar{w}_{m,n} = \begin{cases} w_{m,n}, & 0 \le m \le M-1, \quad 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

To compute the output element v_{m_0,n_0}, $0 \le m_0 \le M-1$, $0 \le n_0 \le N-1$, we put the center of the mask (i.e., $t_{0,0}$) on $\bar{w}_{m_0,n_0}$, multiply $\bar{w}_{m,n}$ with the corresponding coefficients t_{m_0-m,n_0-n}, and sum the resulting products. Now let us use the mask (2.6) to operate on a periodic sequence

$$\tilde{w}_{m,n} = w_{m \bmod M, n \bmod N}, \qquad -\infty < m < \infty, \quad -\infty < n < \infty. \tag{2.8}$$

This defines a block circulant matrix-vector product $R\mathbf{w}$, which is close to the operation $T\mathbf{w}$. Since $R^{-1}\mathbf{v}$ can be computed efficiently with two-dimensional fast Fourier transform, it is natural to use R as a preconditioner for T.

The characterization of a block Toeplitz or circulant matrix by a two-dimensional operator mask is not new. It is basically the same as the stencil form used in the finite-difference discretization of the constant-coefficient partial differential operator. For example, the five-point stencil discretization of the Poisson equation can be interpreted as the mask

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

operating on a two-dimensional sequence. By assuming the Dirichlet and periodic boundary conditions, we obtain block Toeplitz and circulant matrices, respectively.

The preconditioner constructed above can be described in matrix notation. First, for every point Toeplitz matrix T_n, $|n| \le N-1$, we construct a circulant preconditioner $\mathcal{R}_n$ with

$$t_{0,n}, \ t_{-1,n} + t_{M-1,n}, \ t_{-2,n} + t_{M-2,n}, \ \cdots, \ t_{1-M,n} + t_{1,n} \tag{2.9}$$

as the first row [10]. Then, we use $\mathcal{R}_n$ to construct R_n according to the linear combination

$$R_n = \begin{cases} \mathcal{R}_n, & n = 0, \\ \mathcal{R}_n + \mathcal{R}_{N-n}, & 1 \le n \le N-1, \end{cases} \tag{2.10}$$

which is used in (2.3) to define the block circulant preconditioner R.

It is worthwhile to point out that it is possible to design different preconditioners by considering different periodic extensions to form $\tilde{w}_{m,n}$. For readers interested in the design of preconditioners, we refer to [10].

3. The spectral clustering property of preconditioned block Toeplitz matrices. Let us consider a family of block Toeplitz matrices T whose generating sequences $t_{m,n}$ are quadrantally-symmetric,

$$t_{m,n} = t_{|m|,|n|}, \quad |m| \le M-1, \quad |n| \le N-1, \tag{3.1}$$

and absolutely summable (i.e., T is in the Wiener class),

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |t_{m,n}| \le K < \infty, \tag{3.2}$$

and whose generating functions are of the form

$$T(z_x, z_y) \equiv \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} t_{i,j} z_x^{-i} z_y^{-j}$$

$$= \frac{A(z_x, z_y)}{B(z_x, z_y)} + \frac{A(z_x^{-1}, z_y)}{B(z_x^{-1}, z_y)} + \frac{A(z_x, z_y^{-1})}{B(z_x, z_y^{-1})} + \frac{A(z_x^{-1}, z_y^{-1})}{B(z_x^{-1}, z_y^{-1})}, \tag{3.3a}$$

where

$$A(z_x, z_y) = \sum_{i=0}^{p_x}\sum_{j=0}^{p_y} a_{i,j} z_x^{-i} z_y^{-j}, \qquad B(z_x, z_y) = \sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j} z_x^{-i} z_y^{-j}. \tag{3.3b}$$

Note that the quadrantally-symmetric property of $t_{m,n}$ implies that T is doubly symmetric, i.e., $T_n = T_n^T$ and $T_n = T_{-n}$. We also assume that T has a nonsingular preconditioner R so that $R^{-1}T$ is also nonsingular. We call T satisfying (3.1)–(3.3) the $MN \times MN$ block Toeplitz matrix generated by a quadrantally-symmetric rational function of order (p_x, q_x, p_y, q_y). For convenience, we use the notation

$$\gamma_x = \max(p_x, q_x), \qquad \gamma_y = \max(p_y, q_y).$$

The following discussion focuses on the spectral clustering property of the preconditioned matrix $R^{-1}T$, namely, a bound on the number of eigenvalues clustered around unity. Note that the following spectral analysis does not depend on the positive-definiteness of T or R.

3.1. Separable generating sequences. One special case of block Toeplitz matrices described by (3.1)–(3.3) is that $T(z_x, z_y)$ is separable, i.e.,

$$T(z_x, z_y) = T_x(z_x) T_y(z_y), \tag{3.4a}$$

where

$$T_x(z_x) = \frac{A_x(z_x)}{B_x(z_x)} + \frac{A_x(z_x^{-1})}{B_x(z_x^{-1})}, \qquad T_y(z_y) = \frac{A_y(z_y)}{B_y(z_y)} + \frac{A_y(z_y^{-1})}{B_y(z_y^{-1})}, \tag{3.4b}$$

and where

(3.4c)
$$A_x(z_x) = \sum_{i=0}^{p_x} a_i z_x^{-i}, \quad B_x(z_x) = \sum_{i=0}^{q_x} b_i z_x^{-i}, \quad A_y(z_y) = \sum_{i=0}^{p_y} c_i z_y^{-i}, \quad B_y(z_y) = \sum_{i=0}^{q_y} d_i z_y^{-i}.$$

Note that the separability of $T(z_x, z_y)$ implies the separability of the generating sequence $t_{m,n}$, i.e.,

$$t_{m,n} = t_{x,m} t_{y,n}.$$

Based on $t_{x,m}$ (or $t_{y,n}$), we can construct Toeplitz matrix T_x (or T_y) and the corresponding preconditioner R_x (or R_y), where T_x and R_x (or T_y and R_y) are of dimension $M \times M$ (or $N \times N$). It is easy to see that the preconditioner R is also separable, and the eigenvalues of $R^{-1}T$ are the products of the eigenvalues of $R_x^{-1}T_x$ and $R_y^{-1}T_y$.

Thus, to understand the spectral properties of $R^{-1}T$, we only have to examine those of preconditioned (point) Toeplitz matrices $R_x^{-1}T_x$ and $R_y^{-1}T_y$.

According to the construction (2.9) and the symmetric property of $t_{x,m}$, we know that R_x is a circulant matrix with the first row

$$t_{x,0},\ t_{x,1}+t_{x,M-1},\ t_{x,2}+t_{x,M-2},\ \cdots,\ t_{x,M-1}+t_{x,1}.$$

When $B_x(z_x)=1$ (i.e., $q_x=0$), T_x is banded with bandwidth p_x, and R_x is almost the same as T_x except for the addition of elements in the northeast and southwest corners to make R_x circulant. Thus, the elements of $\triangle T_x = R_x - T_x$ are all zeros except the first and the last p_x rows. Consequently, $\triangle T_x$ has at least $M-2p_x$ $(=M-2\gamma_x)$ eigenvalues at zero and $R_x^{-1}T_x=(T_x+\triangle T_x)^{-1}T_x$ has at least $M-2p_x$ eigenvalues at one. This result can also be obtained by using the operator-mask interpretation. That is, the products $T_x\mathbf{w}$ and $R_x\mathbf{w}$, for arbitrary $\mathbf{w}=(w_0,\cdots,w_m,\cdots,w_{M-1})^T$, can be viewed as a linear operator characterized by the mask

$$[t_{x,p_x}\ \ t_{x,p_x-1}\ \ \cdots\ \ t_{x,1}\ \ t_{x,0}\ \ t_{x,1}\ \ \cdots\ \ t_{x,p_x-1}\ \ t_{x,p_x}]$$

operating, respectively, on two extended sequences

$$\bar{w}_m=\begin{cases} w_m, & 0\le m\le M-1,\\ 0, & \text{otherwise,}\end{cases} \quad \text{and} \quad \tilde{w}_m = w_{m \bmod M}.$$

It is clear that $T_x\mathbf{w}$ and $R_x\mathbf{w}$ give the same output elements if the center of the mask is located at $p_x\le m\le M-p_x-1$. There are $M-2p_x$ such elements and, as a consequence, the dimension of the null space of $\triangle T_x$ is at least $M-2p_x$. The operator-mask viewpoint will be generalized to the case of higher-dimensional generating sequences (see §3.2).

When $B_x(z_x)\neq 1$, we approximate the block matrix $\triangle T_x = R_x - T_x$ with an asymptotically equivalent block matrix $\triangle E_x$, and then use the recursive property of $t_{x,m}$ to show that $\triangle E_x$ has eigenvalues repeated at zero. The recursive property of $t_{x,m}$ is stated in the following lemma.

LEMMA 1. *The sequence $t_{x,m}$ generated by $T_x(z_x)$ in* (3.4b) *follows the recursion,*

(3.5)
$$t_{x,m+1}=-(b_1t_{x,m}+b_2t_{x,m-1}+\cdots+b_{q_x}t_{x,m-q_x+1})/b_0,\quad m\ge\gamma_x=\max(p_x,q_x).$$

Proof. The generating sequence associated with $A_x(z_x)/B_x(z_x)$ given by (3.4c) is

$$\tfrac{1}{2}t_{x,0},t_{x,1},t_{x,2},\cdots,t_{x,m},\cdots.$$

Thus, we have

$$\left(\frac{t_{x,0}}{2}+\sum_{m=1}^{\infty}t_{x,m}z^{-m}\right)(b_0+b_1z^{-1}+\cdots+b_{q_x}z^{-q_x})=a_0+a_1z^{-1}+\cdots+a_{p_x}z^{-p_x}.$$

The proof is completed by comparing the coefficients of the above equation. $\square$

Consider the approximation of $\triangle T_x = R_x - T_x$ with $\triangle E_x = F_x + F_x^T$, where

$$F_x=\begin{bmatrix} t_M & t_{M-1} & \cdot & \cdot & t_2 & t_1\\ t_{M+1} & t_M & t_{M-1} & \cdot & \cdot & t_2\\ \cdot & t_{M+1} & t_M & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & t_{M-1} & \cdot\\ t_{2M-2} & \cdot & \cdot & t_{M+1} & t_M & t_{M-1}\\ t_{2M-1} & t_{2M-2} & \cdot & \cdot & t_{M+1} & t_M \end{bmatrix},$$

and where the subscript x is omitted for simplicity and t_m with $m > \gamma_x$ is recursively constructed from (3.5). Since elements $t_{x,m}$ in F_x satisfy (3.5), the rank of F_x or F_x^T is at most γ_x. Consequently, $\triangle E_x$ has at least $M - 2\gamma_x$ eigenvalues at zero.

Then, we examine the difference between $\triangle T_x$ and $\triangle E_x$. Consider

$$\mathbf{v} = (\triangle T_x - \triangle E_x)\mathbf{w}, \tag{3.6}$$

for nonzero $\mathbf{w}$. The mth, $1 \le m \le M$, elements of $\triangle T_x \mathbf{w}$ and $\triangle E_x \mathbf{w}$ can be written, respectively, as

$$[\triangle T_x \mathbf{w}]_m = [R_x \mathbf{w}]_m - [T_x \mathbf{w}]_m = \sum_{i=m+1}^{M} t_{x,M+m-i} w_i + \sum_{i=1}^{m-1} t_{x,M-m+i} w_i, \tag{3.7}$$

and

$$[\triangle E_x \mathbf{w}]_m = [F_x \mathbf{w}]_m + [F_x^T \mathbf{w}]_m = \sum_{i=1}^{M} t_{x,M+m-i} w_i + \sum_{i=1}^{M} t_{x,M-m+i} w_i. \tag{3.8}$$

Therefore, $\mathbf{v}$ is bounded above by

$$\begin{aligned} ||\mathbf{v}||_\infty = ||\triangle E_x \mathbf{w} - \triangle T_x \mathbf{w}||_\infty &\le \max_{1 \le m \le M} \left| \sum_{i=1}^{m} t_{x,M+m-i} w_i + \sum_{i=m}^{M} t_{x,M-m+i} w_i \right| \\ &\le 2 \sum_{m=M}^{2M-1} |t_{x,m}| ||\mathbf{w}||_\infty, \end{aligned}$$

where the ∞-norm of the vector $\mathbf{v}$ (or $\mathbf{w}$) is the maximum absolute value of elements v_m (or w_m), $1 \le m \le M$. We have

$$||\triangle T_x - \triangle E_x||_\infty \equiv \max_{\mathbf{w}} \frac{||\mathbf{v}||_\infty}{||\mathbf{w}||_\infty} \le 2 \sum_{m=M}^{2M-1} |t_{x,m}|, \tag{3.9}$$

which converges to zero as M goes to infinity, since $\sum_{m=0}^{\infty} |t_{x,m}|$ converges and $S_m = \sum_{m'=0}^{m} |t_{x,m'}|$ is a Cauchy sequence. The matrix $\triangle T_x - \triangle E_x$ is symmetric so that

$$||\triangle T_x - \triangle E_x||_1 = ||\triangle T_x - \triangle E_x||_\infty$$

and

$$||\triangle T_x - \triangle E_x||_2 \le (||\triangle T_x - \triangle E_x||_1 ||\triangle T_x - \triangle E_x||_\infty)^{1/2} = ||\triangle T_x - \triangle E_x||_\infty.$$

Thus, $\triangle T_x$ is asymptotically equivalent to $\triangle E_x$. It also follows that $\triangle T_x$ has at least $M - 2\gamma_x$ eigenvalues asymptotically converging to 0, or $R_x^{-1} T_x = (T_x + \triangle T_x)^{-1} T_x$ has at least $M - 2\gamma_x$ eigenvalues asymptotically converging to 1.

The $\triangle T_x$ and $\triangle E_x$ above are amenable to the operator-mask interpretation (see Fig. 1 with $M = 8$). One can easily verify that $\triangle T_x \mathbf{w}$ and $\triangle E_x \mathbf{w}$ correspond, respectively, to the use of the two masks

$$\begin{aligned} &\triangle T_x : [t_{x,M-1}\ t_{x,M-2}\ \cdots\ t_{x,1}\ t_{x,0}\ t_{x,1}\ \cdots\ t_{x,M-2}\ t_{x,M-1}], \\ &\triangle E_x : [\cdots\ t_{x,M}\ t_{x,M-1}\ \cdots\ t_{x,1}\ t_{x,0}\ t_{x,1}\ \cdots\ t_{x,M-1}\ t_{x,M}\ \cdots], \end{aligned}$$

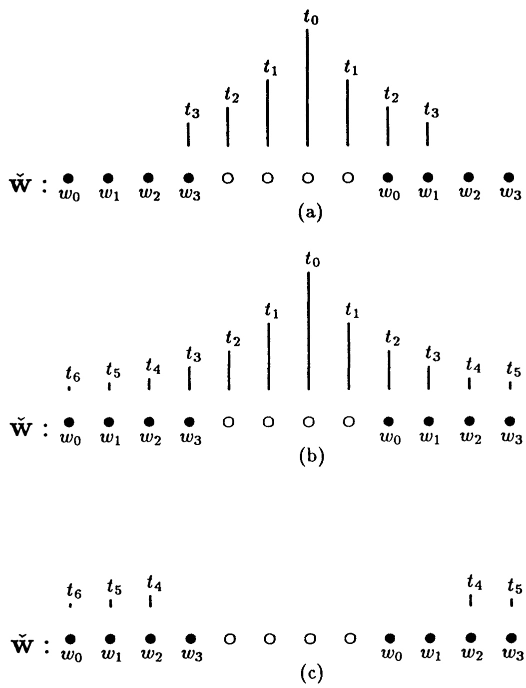


FIG. 1. *Operator-mask interpretations of* (a) $\triangle T_x \mathbf{w}$, (b) $\triangle E_x \mathbf{w}$, *and* (c) $(\triangle E_x - \triangle T_x)\mathbf{w}$.

operating on the same sequence

$$\breve{w}_m = \begin{cases} w_{m \bmod M}, & -M \le m \le -1 \quad \text{or} \quad M \le m \le 2M-1, \\ 0, & \text{elsewhere.} \end{cases}$$

Note that the mask for $\triangle E_x$ is of infinite length. The corresponding mask for $\triangle E_x - \triangle T_x$ is

$$\triangle E_x - \triangle T_x : [\cdots \; t_{x,M+1} \; t_{x,M} \; 0 \; \cdots \; 0 \; 0 \; 0 \; \cdots \; 0 \; t_{x,M} \; t_{x,M+1} \; \cdots].$$

It is easy to derive (3.9) from the operator-mask viewpoint. Note that, for larger M, although there are more terms contributing to the ∞-norm of $\triangle E_x - \triangle T_x$, the weighting coefficient $t_{x,m}$, $m \geq M$, decays more rapidly. The resulting ∞-norm of $\triangle E_x - \triangle T_x$ asymptotically converges to zero. We conclude the above discussion as follows.

(a) When $B(z_x, z_y) = 1$, T_x (or T_y) is banded and $R_x^{-1}T_x$ (or $R_y^{-1}T_y$) has at most $2p_x+1$ (or $2p_y+1$) distinct eigenvalues, so that $R^{-1}T$ has at most $(2p_x+1)(2p_y+1)$ distinct eigenvalues.

(b) When $B(z_x, z_y) \neq 1$, $R_x^{-1}T_x$ (or $R_y^{-1}T_y$) has at most $2\gamma_x$ (or $2\gamma_y$) outliers not converging to unity and other eigenvalues are clustered between $(1-\epsilon_x, 1+\epsilon_x)$ (or $(1-\epsilon_y, 1+\epsilon_y)$). Thus, the eigenvalues of $R^{-1}T$ can be grouped into several clusters. The centers and clustering radii of these clusters and the numbers of eigenvalues contained are listed in Table 1, where $\lambda_{x,i}$ (or $\lambda_{y,j}$) denotes a typical outlier for

TABLE 1
Eigenvalues of $R^{-1}T$.

Center	$\lambda_{x,i}\lambda_{y,j}$	$\lambda_{x,i}$	$\lambda_{y,j}$	1
Radius	0	$O(\epsilon_y)$	$O(\epsilon_x)$	$O(\epsilon_x+\epsilon_y)$
Number	1	$N-2\gamma_y$	$M-2\gamma_x$	$(M-2\gamma_x)(N-2\gamma_y)$

$R_x^{-1}T_x$ (or $R_y^{-1}T_y$). Since ϵ_x and ϵ_y converge to zero as M and N become large, $R^{-1}T$ has asymptotically at most $(2\gamma_x+1)(2\gamma_y+1)$ distinct eigenvalues.

As a consequence, the PCG method converges in at most $(2\gamma_x+1)(2\gamma_y+1)$ iterations for positive definite T with sufficiently large M and N in both cases (a) and (b). This is confirmed numerically in §4.

3.2. Nonseparable generating sequences. For nonseparable generating functions $T(z_x,z_y)$ given by (3.3), we examine two typical cases, i.e., $B(z_x,z_y)=1$ and $B(z_x,z_y)\neq 1$ with $q_x>0$ and $q_y>0$.

When $B(z_x,z_y)=1$, we have a corresponding generating sequence of finite duration. As described in §2, the products $T\mathbf{w}$ and $R\mathbf{w}$ correspond to a linear operator characterized by the mask

$$\begin{bmatrix} t_{p_x,p_y} & t_{p_x-1,p_y} & \cdots & t_{0,p_y} & \cdots & t_{p_x-1,p_y} & t_{p_x,p_y} \\ t_{p_x,p_y-1} & t_{p_x-1,p_y-1} & \cdots & t_{0,p_y-1} & \cdots & t_{p_x-1,p_y-1} & t_{p_x,p_y-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{p_x,0} & t_{p_x-1,0} & \cdots & t_{0,0} & \cdots & t_{p_x-1,0} & t_{p_x,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{p_x,p_y-1} & t_{p_x-1,p_y-1} & \cdots & t_{0,p_y-1} & \cdots & t_{p_x-1,p_y-1} & t_{p_x,p_y-1} \\ t_{p_x,p_y} & t_{p_x-1,p_y} & \cdots & t_{0,p_y} & \cdots & t_{p_x-1,p_y} & t_{p_x,p_y} \end{bmatrix}$$

operating, respectively, on $\bar{w}_{m,n}$ and $\tilde{w}_{m,n}$ given by (2.7) and (2.8). The output elements of $T\mathbf{w}$ and $R\mathbf{w}$ are identical if the center of the mask is located at

$$p_x \le m \le M-p_x-1, \qquad p_y \le n \le N-p_y-1.$$

The dimension of the null space of $\triangle T = R-T$ is at least $(M-2p_x)(N-2p_y)$. Consequently, $R^{-1}T$ has at least $(M-2p_x)(N-2p_y)$ eigenvalues repeated at 1 or, equivalently, there are at most $2(Mp_y+Np_x)-4p_xp_y$ outliers. This result is summarized in the following theorem.

THEOREM 1. *Let T be an $MN\times MN$ block Toeplitz matrix characterized by (3.1)–(3.3) with $B(z_x,z_y)=1$. The preconditioned matrix $R^{-1}T$ has at least $MN-2(Mp_y+Np_x)+4p_xp_y$ eigenvalues repeated at one.*

When $B(z_x,z_y)\neq 1$ with $q_x>0$ and $q_y>0$, the products $T\mathbf{w}$ and $R\mathbf{w}$ correspond to a linear operator characterized by the mask

$$\begin{bmatrix} t_{M-1,N-1} & t_{M-2,N-1} & \cdots & t_{0,N-1} & \cdots & t_{M-2,N-1} & t_{M-1,N-1} \\ t_{M-1,N-2} & t_{M-2,N-2} & \cdots & t_{0,N-2} & \cdots & t_{M-2,N-2} & t_{M-1,N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,0} & t_{M-2,0} & \cdots & t_{0,0} & \cdots & t_{M-2,0} & t_{M-1,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,N-2} & t_{M-2,N-2} & \cdots & t_{0,N-2} & \cdots & t_{M-2,N-2} & t_{M-1,N-2} \\ t_{M-1,N-1} & t_{M-2,N-1} & \cdots & t_{0,N-1} & \cdots & t_{M-2,N-1} & t_{M-1,N-1} \end{bmatrix} \tag{3.10}$$

operating on $\bar{w}_{m,n}$ and $\tilde{w}_{m,n}$, respectively. Let us exploit the quadrantally-symmetric property (3.1) and decompose $t_{m,n}$ into four sequences

$$t_{m,n} = \sum_{k=1}^{4} t_{k,m,n},$$

where $t_{k,m,n}$ is called the kth quadrant-support sequence and defined as

$$t_{1,m,n} = \begin{cases} t_{0,0}/4, & m = n = 0, \\ t_{m,0}/2, & 1 \le m \le M-1, \quad n = 0, \\ t_{0,n}/2, & m = 0, \quad 1 \le n \le N-1, \\ t_{m,n}, & 1 \le m \le M-1 \quad \text{and} \quad 1 \le n \le N-1, \\ 0, & m < 0 \quad \text{or} \quad n < 0, \end{cases} \tag{3.11a}$$

and

$$t_{2,m,n} = t_{1,-m,n}, \qquad t_{3,m,n} = t_{1,-m,-n}, \qquad t_{4,m,n} = t_{1,m,-n}. \tag{3.11b}$$

The following lemma is on the recursive property of $t_{1,m,n}$.

LEMMA 2. *Let $t_{m,n}$ be a quadrantally-symmetric sequence generated by the two-dimensional rational function $T(z_x, z_y)$ given in* (3.3), *and $t_{1,m,n}$ is the first quadrant-support sequence defined by* (3.11a). *Then,*

$$\sum_{i=0}^{q_x} \sum_{j=0}^{q_y} b_{i,j} t_{1,m-i,n-j} = 0 \qquad \text{for } m > \gamma_x \quad \text{or} \quad n > \gamma_y. \tag{3.12}$$

Proof. It is clear that $A(z_x, z_y)/B(z_x, z_y)$ is the generating function for $t_{1,m,n}$. Therefore,

$$\frac{A(z_x, z_y)}{B(z_x, z_y)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_{1,i,j} z_x^{-i} z_y^{-j}.$$

We multiply both sides of the above equation with $B(z_x, z_y)$, substitute (3.3b) for $A(z_x, z_y)$ and $B(z_x, z_y)$, and compare the corresponding coefficients. This gives the desired equation (3.12). $\square$

Thus, we can use (3.12) to recursively define $t_{1,m,n}$ with $m > \gamma_x$ or $n > \gamma_y$ in the first quadrant, and the corresponding $t_{k,m,n}$ with $k = 2, 3, 4$ can be obtained from $t_{1,m,n}$ through the symmetry (3.11b).

As a generalization of the one-dimensional case, we define

$$\breve{w}_{m,n} = \begin{cases} w_{m \bmod M, n \bmod N}, & (m,n) \in [\cup_{-1 \le i,j \le 1} Q_{i,j}] - Q_{0,0}, \\ 0, & \text{elsewhere}, \end{cases}$$

where

$$Q_{i,j} = \{(m,n): \quad iM \le m \le (i+1)M - 1, \quad jN \le n \le (j+1)N - 1\}.$$

Then, the operation $\triangle T\mathbf{w} = (R - T)\mathbf{w}$ corresponds to the mask (3.10) operating on $\breve{w}_{m,n}$. We choose the approximation $\triangle E\mathbf{w}$ to be an extended infinite mask, with recursively defined $t_{m,n}$ via (3.12), operating on $\breve{w}_{m,n}$. This is illustrated in Fig. 2, where we only show the first quadrant of the mask for $\triangle E$. For the rest of this

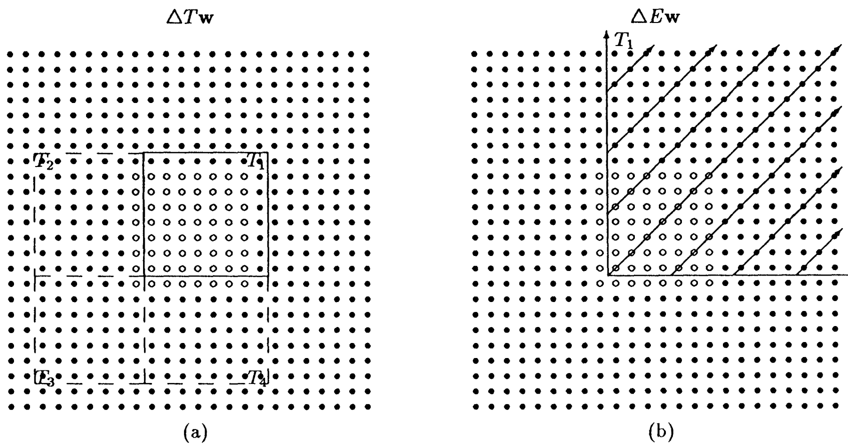


FIG. 2. *The matrix vector products* $\triangle T\mathbf{w}$ *and* $\triangle E\mathbf{w}$ *interpreted as* (a) *the operator-mask and* (b) *the extended operator-mask operating on* $\breve{w}$.

subsection, we are concerned with two issues: (i) the asymptotic equivalence of $\triangle T$ and $\triangle E$ and (ii) the number of eigenvalues of $\triangle E$ repeated at zero.

The operation $(\triangle E - \triangle T)\mathbf{w}$ corresponds to the difference between the extended mask and the original mask (3.10) operating on $\breve{w}_{m,n}$. The ∞-norm of the first quadrant of the difference mask $\triangle E - \triangle T$ operating on $\breve{w}_{m,n}$ in regions $Q_{1,0}$, $Q_{0,1}$, and $Q_{1,1}$ are bounded, respectively, by

$$K_{1,1,0} = \sum_{m=M}^{2M-1} \sum_{n=0}^{N-1} |t_{1,m,n}|, \quad K_{1,0,1} = \sum_{m=0}^{M-1} \sum_{n=N}^{2N-1} |t_{1,m,n}|, \quad K_{1,1,1} = \sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} |t_{1,m,n}|.$$

By exploiting the symmetry, we have the bound for $\triangle E - \triangle T$,

$$||\triangle E - \triangle T||_\infty \le 4(K_{1,1,0} + K_{1,0,1} + K_{1,1,1}) = K.$$

With (3.2), we can order $t_{m,n}$ appropriately to be a Cauchy sequence and argue that K converges to zero for asymptotically large M and N. This establishes the asymptotic equivalence of $\triangle E$ and $\triangle T$.

The operator $\triangle E$ can be expressed as a superposition of 12 operators,

$$\begin{aligned} \triangle E = {} & F_{1,1,0} + F_{1,1,1} + F_{1,0,1} + F_{2,0,1} + F_{2,-1,1} + F_{2,-1,0}, \\ & + F_{3,-1,0} + F_{3,-1,-1} + F_{3,0,-1} + F_{4,0,-1} + F_{4,1,-1} + F_{4,1,0}, \end{aligned} \tag{3.13}$$

where $F_{k,i,j}$ denotes the kth quadrant of the extended mask operating on sequences defined on $Q_{i,j}$. Consider operators $F_{1,1,0}$, $F_{1,0,1}$, and $F_{1,1,1}$. Their operations on $\mathbf{w}$ can be written as

$$F_{1,1,0} \ : \ v_{1,i,j} = \sum_{m=0}^{M-1} \sum_{n=j}^{N-1} t_{1,M+m-i,n-j} w_{m,n},$$

$$F_{1,0,1} \ : \ v_{2,i,j} = \sum_{m=i}^{M-1} \sum_{n=0}^{N-1} t_{1,m-i,N+n-j} w_{m,n},$$

$$F_{1,1,1} \ : \ v_{3,i,j} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} t_{1,M+m-i,N+n-j} w_{m,n}.$$

With (3.12), we have

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0} v_{1,i,j} = \sum_{m=0}^{M-1}\sum_{n=j+j_0}^{N-1}\left[\sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j} t_{1,M+m-i_0-i,n-j_0-j}\right] w_{m,n} = 0,$$

for $0 \le i_0 \le M-1-\gamma_x$;

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0} v_{2,i,j} = \sum_{m=i+i_0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j} t_{1,m-i_0-i,N+n-j_0-j}\right] w_{m,n} = 0,$$

for $0 \le j_0 \le N-1-\gamma_y$; and

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0} v_{3,i,j} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j} t_{1,M+m-i_0-i,N+n-j_0-j}\right] w_{m,n} = 0,$$

for $0 \le i_0 \le M-1-\gamma_x$ or $0 \le j_0 \le N-1-\gamma_y$.

By combining the above three equations, we have

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0}(v_{1,i,j}+v_{2,i,j}+v_{3,i,j}) = 0,$$

for $0 \le i_0 \le M-1-\gamma_x$ and $0 \le j_0 \le N-1-\gamma_y$. Therefore, the rank of $F_1 = F_{1,1,0}+F_{1,0,1}+F_{1,1,1}$ is at most $M\gamma_y + N\gamma_x - \gamma_x\gamma_y$. By using the symmetry, we can argue that the rank of $F_k = \sum_{i,j} F_{k,i,j}$, $k = 2,3,4$, is also at most $M\gamma_y + N\gamma_x - \gamma_x\gamma_y$. Consequently, the rank of $\triangle E$ is at most $4(M\gamma_y + N\gamma_x - \gamma_x\gamma_y)$ or, equivalently, $\triangle E$ has at least $MN - 4(M\gamma_y + N\gamma_x - \gamma_x\gamma_y)$ eigenvalues repeated at 0.

To conclude this section, we have the following theorem.

THEOREM 2. *Let T be an $MN \times MN$ block Toeplitz matrix satisfying* (3.1), (3.2), *and* (3.3) *with $q_x > 0$ and $q_y > 0$. Then, the preconditioned matrix $R^{-1}T$ has at least $MN - 4(M\gamma_x + N\gamma_y - \gamma_x\gamma_y)$ eigenvalues asymptotically converging to one.*

4. Numerical experiments. Numerical experiments are performed to illustrate the spectra of T and $R^{-1}T$ and the convergence behavior of the CG and PCG methods. Note that the spectral clustering property derived in §3 does not require T to be positive definite. However, we focus on positive-definite T in our experiments so that the CG and PCG methods can be conveniently applied. For all test problems below, we choose $M = N$ and use $(0, \cdots, 0)^T$ and $(1, \cdots, 1)^T$ as the initial and right-hand-side vectors, respectively.

The first three test problems have positive rational generating functions in the Wiener class.

Example 1. Rational separable Toeplitz with $(p_x, q_x, p_y, q_y) = (0, 2, 0, 2)$. The $T(z_x, z_y)$ is of the form (3.4) with $A_x(z_x) = A_y(z_y) = 1$,

$$B_x(z_x) = (1+0.8z_x^{-1})(1-0.7z_x^{-1}) \quad \text{and} \quad B_y(z_y) = (1+0.9z_y^{-1})(1-0.6z_y^{-1}).$$

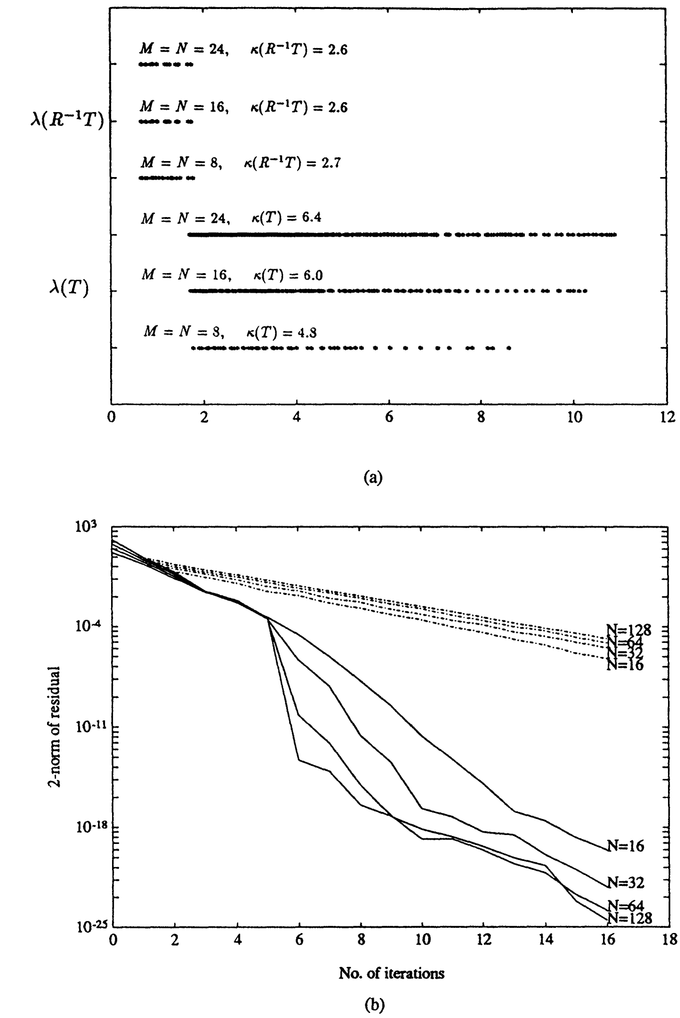


FIG. 3. (a) *Eigenvalue distribution of* T *and* $R^{-1}T$ *and* (b) *convergence history for Example* 1.

Example 2. Rational Toeplitz with $(p_x, q_x, p_y, q_y) = (2, 0, 2, 0)$. The $T(z_x, z_y)$ is of the form (3.3) with $B(z_x, z_y) = 1$ and

$$A(z_x, z_y) = 0.25 - 0.02(z_x^{-1} + z_y^{-1}) + 0.015(z_x^{-2} + z_y^{-2}) + 0.03z_x^{-1}z_y^{-1}$$
$$-0.02z_x^{-1}z_y^{-1}(z_x^{-1} + z_y^{-1}) - 0.01z_x^{-2}z_y^{-2}.$$

Example 3. Rational Toeplitz with $(p_x, q_x, p_y, q_y) = (0, 2, 0, 1)$. The $T(z_x, z_y)$ is of the form (3.3) with $A(z_x, z_y) = 1$ and

$$B(z_x, z_y) = 1 + 0.5z_x^{-1} - 0.3z_y^{-1} - 0.2z_x^{-2} - 0.1z_x^{-1}z_y^{-1} + 0.2z_x^{-2}z_y^{-1}.$$

We plot the corresponding spectra of T and $R^{-1}T$, their condition numbers, and the convergence history of the CG (dashed lines) and PCG (solid lines) methods in Figs. 3(a),(b)–5(a),(b). Since the eigenvalues of T for Examples 1–3 all satisfy

$$0 < \delta_1 \leq \lambda(T) \leq \delta_2 < \infty,$$

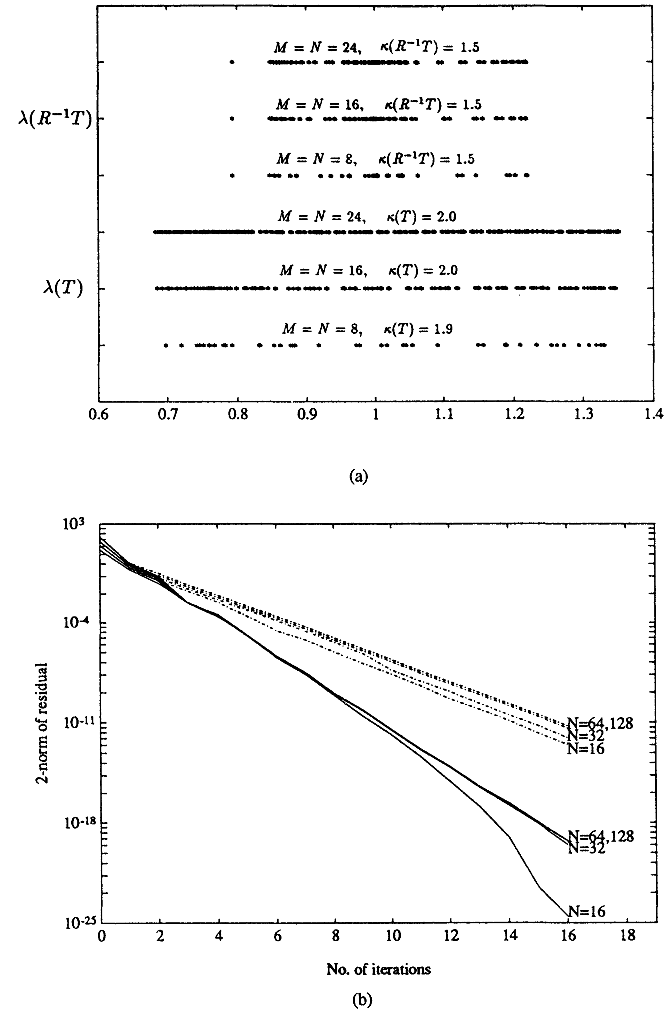
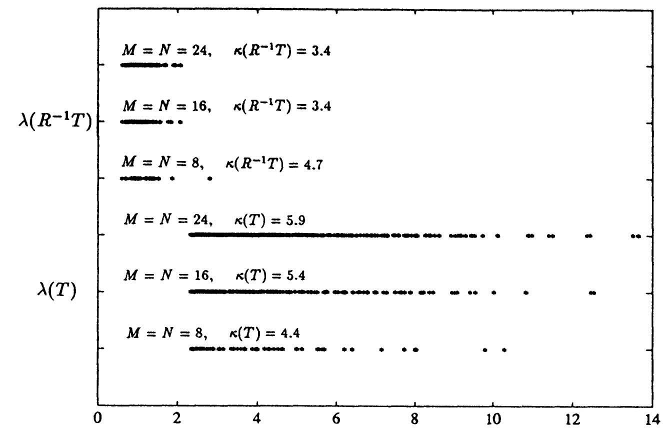


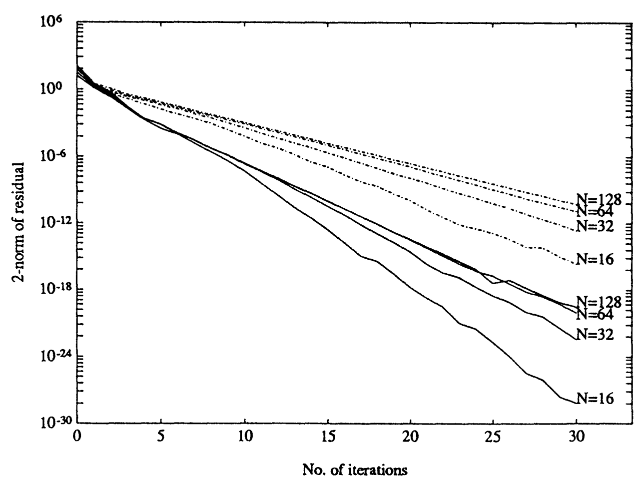
FIG. 4. (a) *Eigenvalue distribution of* T *and* $R^{-1}T$ *and* (b) *convergence history for Example* 2.

where δ_1 and δ_2 are constants independent of the dimensions M and N of the given block matrix, the condition number $\kappa(T)$ is bounded by $\delta_2/\delta_1 = O(1)$. Clearly, $R^{-1}T$ has a smaller condition number and a better clustering feature than T. Consequently, the PCG method performs better than the CG method.

One important difference between the separable and nonseparable cases is that, as N becomes larger, the PCG method converges faster for the separable case but more slowly for the nonseparable case. This can be easily explained by the analysis given in §3. When T is separable, the number of clusters is fixed and the clustering radius ϵ becomes smaller as N becomes larger. According to the analysis, $R^{-1}T$ has asymptotically 25 $(= (2\gamma_x + 1)(2\gamma_y + 1))$ distinct eigenvalues, including isolated outliers, clustered outliers, and clustered eigenvalues converging to 1. However, the 2-norm of the residual decreases rapidly in five iterations as given in Fig. 3(b). We observe an empirical formula for the separable case, namely, the PCG method converges



(a)



(b)

FIG. 5. (a) *Eigenvalue distribution of* T *and* $R^{-1}T$ *and* (b) *convergence history for Example* 3.

asymptotically in $\gamma_x\gamma_y + 1$ iterations. This phenomenon is closely related to the point Toeplitz result [10], where we found that although there are asymptotically $2\gamma_x + 1$ distinct eigenvalues, the PCG method converges asymptotically in $\gamma_x + 1$ iterations. When T is nonseparable, the number of outliers increases with N. Although the PCG method converges more slowly for larger N, the effect is not obvious until the 2-norm of the residual is very small. Besides, the convergence curves are getting closer for larger N. This indicates that the number of PCG (or CG) iterations required is $O(1)$, which is determined by the condition number rather than the number of outliers.

A block Toeplitz with a nonrational generating function is given in Example 4.

Example 4. Nonrational Toeplitz. The block Toeplitz matrix is generated by a spherically-symmetric sequence

$$t_{m,n} = 0.7^{\sqrt{m^2+n^2}} + 0.5^{\sqrt{m^2+n^2}} + 0.3^{\sqrt{m^2+n^2}}.$$

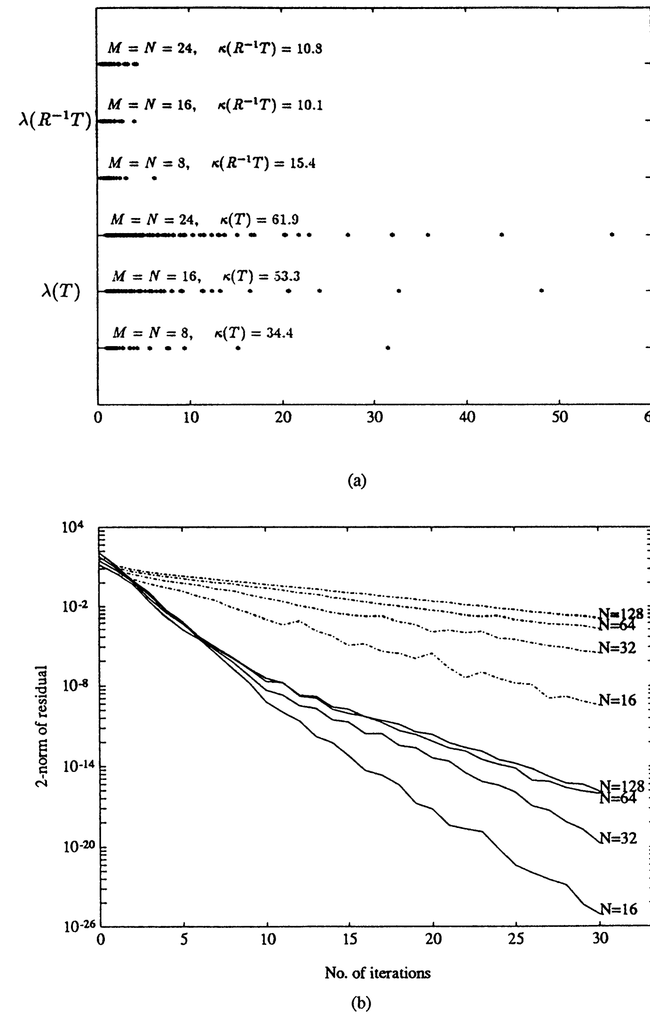


FIG. 6. (a) *Eigenvalue distribution of* T *and* $R^{-1}T$ *and* (b) *convergence history for Example* 4.

The spectra of T and $R^{-1}T$ and the convergence history of the CG and PCG methods are plotted in Fig. 6. Although our analysis in §3 is restricted to the rational generating function case, it appears that this case does not differ much from the rational case. The PCG method converges faster than the CG method due to a smaller condition number and a better spectral clustering property of $R^{-1}T$. The condition numbers of T and $R^{-1}T$ are again $O(1)$ so that the number of PCG (or CG) iterations required is $O(1)$, which is consistent with the observation that the convergence history curves are getting closer for larger N.

For the above four problems, T is well conditioned, i.e., $\kappa(T) = O(1)$, so that the condition number reduction through preconditioning is just a constant factor. To see a more dramatic condition number improvement, let us consider an ill-conditioned block Toeplitz below.

Example 5. Ill-conditioned Toeplitz with $(p_x, q_x, p_y, q_y) = (2, 0, 2, 0)$. The block

Toeplitz is characterized by the two-dimensional mask:

$$(-1) \times \begin{bmatrix} 0.01 & 0.02 & 0.04 & 0.02 & 0.01 \\ 0.02 & 0.04 & 0.12 & 0.04 & 0.02 \\ 0.04 & 0.12 & -1 & 0.12 & 0.04 \\ 0.02 & 0.04 & 0.12 & 0.04 & 0.02 \\ 0.01 & 0.02 & 0.04 & 0.02 & 0.01 \end{bmatrix}.$$

Note that the sum of the coefficients of the mask is zero. Masks of this nature arise in the discretization of constant-coefficient elliptic partial differential equations. However, the preconditioning block circulant matrix R is singular for this problem, since $R\mathbf{w} = 0$ for $\mathbf{w} = (1, 1, \cdots, 1, 1)^T$. In order to perform the preconditioning properly, we modify the preconditioner R slightly by replacing the zero eigenvalue with the smallest nonzero eigenvalue of R in our experiment.

The spectra of T and $R^{-1}T$, the convergence history of the CG and PCG methods, and the number of iterations required for the 2-norm of the relative residual less than 10^{-12} are plotted in Fig. 7 (a)–(c). As shown in Fig. 7(a), the condition numbers of T and $R^{-1}T$ increase at the rates of $O(N^2)$ and $O(N)$, respectively. Thus, the preconditioning provides an order of condition number improvement. A detailed analysis for the improvement of conditioned number from $O(N^2)$ to $O(N)$ is given in [4]. We see from Fig. 7(c) that PCG and CG methods converge in $O(\sqrt{N})$ and $O(N)$ iterations, respectively.

As far as the computational complexity is concerned, both the PCG and CG methods require $O(N^2 \log N)$ operations for Examples 1–4, where the condition numbers of T and $R^{-1}T$ are $O(1)$. For Example 5, the PCG and CG methods require, respectively, $O(N^{5/2} \log N)$ and $O(N^3 \log N)$ operations. For all above test problems, the PCG and CG methods require much lower computational complexity than the direct method, which requires $O(N^5)$ operations.

5. Conclusion. In this research, we extended the preconditioning technique from point Toeplitz matrices to block Toeplitz matrices. We interpreted the block Toeplitz matrix-vector $T\mathbf{w}$ in terms of a two-dimensional constant-coefficient mask operating on a certain two-dimensional sequence construction based on $\mathbf{w}$. This viewpoint provides a natural way to analyze the spectral clustering property of $R^{-1}T$. For block Toeplitz matrices T generated by two-dimensional rational functions $T(z_x, z_y)$ of order (p_x, q_x, p_y, q_y), we showed that the eigenvalues of $R^{-1}T$ are clustered around unity except at most $O(M\gamma_y + N\gamma_x)$ outliers, where $\gamma_x = \max(p_x, q_x)$ and $\gamma_y = \max(p_y, q_y)$. Furthermore, if T is separable, the outliers are clustered together such that $R^{-1}T$ has at most $(2\gamma_x + 1)(2\gamma_y + 1)$ asymptotic distinct eigenvalues. Thus, $R^{-1}T$ has a better spectral clustering property than T. Additionally, it was shown numerically that $R^{-1}T$ generally has a smaller condition number than T. These two spectral properties explain the superior convergence behavior of the PCG method over the CG method.

For point rational Toeplitz matrices, the number of outliers is often small ($= 2\max(p, q)$) and independent of the size of the problem so that it can be used to characterize the convergence rate of the PCG method. However, for block rational Toeplitz matrices, the number of outliers is proportional to the size of the problem, and is often too large to be useful for characterizing the convergence behavior of the PCG method. Hence, we have to examine both the condition number improvement and the spectral clustering effect. More research on the adaptation of the preconditioning technique to more general classes of block Toeplitz matrices, such as indefinite or

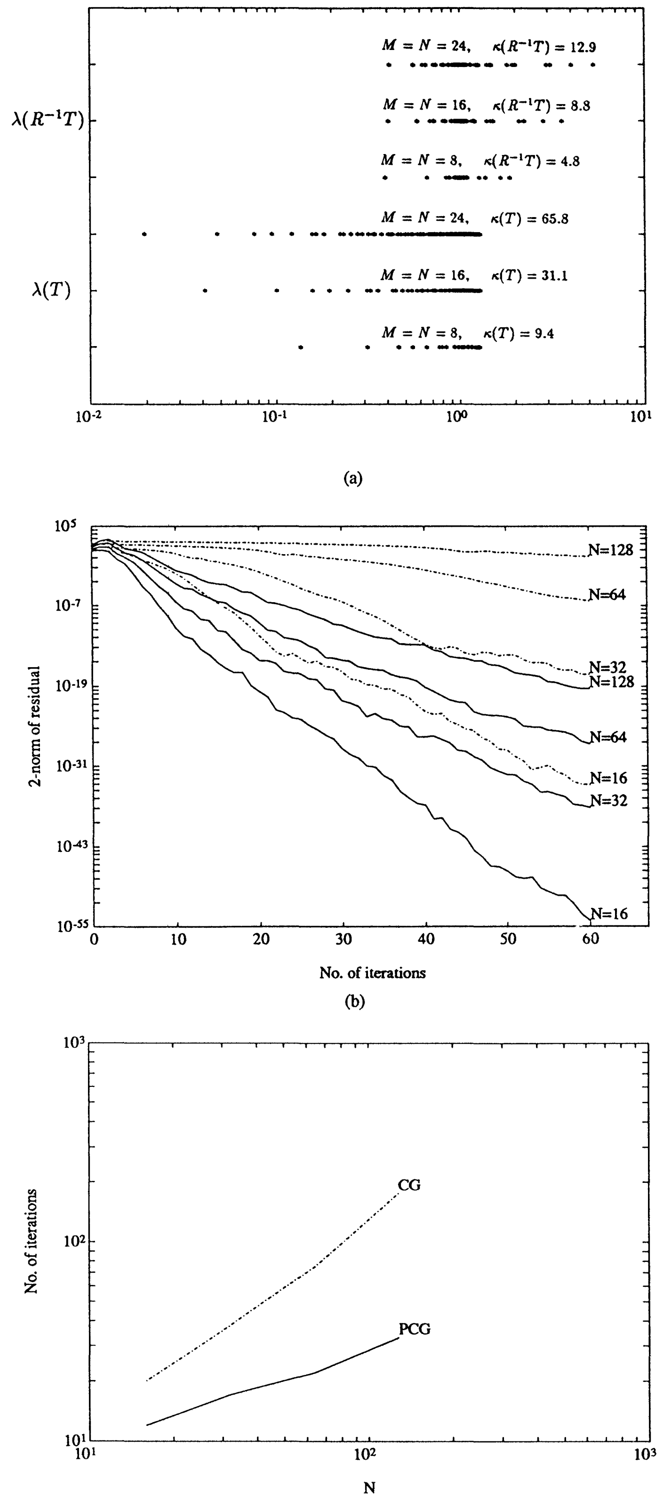


FIG. 7. (a) *Eigenvalue distribution of* T *and* $R^{-1}T$ *and* (b) *convergence history and* (c) *convergence rate of* CG *and* PCG *method for Example* 5.

nonsymmetric problems and the spectral analysis of the preconditioned matrices, is expected in the future.

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