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SPECTRAL PROPERTIES OF PRECONDITIONED RATIONAL TOEPLITZ MATRICES*

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Abstract. Various Toeplitz preconditioners P_N have recently been proposed so that an $N \times N$ symmetric positive definite Toeplitz system $T_N\ \mathbf{x} = \mathbf{b}$ can be solved effectively by the preconditioned conjugate gradient (PCG) method. It has been proven that if T_N is generated by a positive function in the Wiener class, the eigenvalues of the preconditioned matrices $P_N^{-1}T_N$ are clustered between $(1-\epsilon, 1+\epsilon)$ except for a fixed number independent of N. In this research, the spectra of $P_N^{-1}T_N$ are characterized more precisely for rational Toeplitz matrices T_N with preconditioners proposed by Strang [*Stud. Appl. Math.*, 74 (1986), pp. 171–176] and Ku and Kuo [*IEEE Trans. Signal Process.*, 40 (1992), pp. 129–141]. The eigenvalues of $P_N^{-1}T_N$ are classified into two classes, i.e., the outliers and the clustered eigenvalues, depending on whether they converge to 1 asymptotically. It is proved that the number of outliers depends on the order of the rational generating function, and the clustering radius ϵ is proportional to the magnitude of the last element in the generating sequence used to construct these preconditioners. For the special case with T_N generated by a geometric sequence, this approach can be used to determine the exact eigenvalue distribution of $P_N^{-1}T_N$ analytically.

Key words. Toeplitz matrix, preconditioned conjugate gradient method, rational generating function

AMS(MOS) subject classifications. 65F10, 65F15

1. Introduction.

The system of linear equations associated with a symmetric positive definite (SPD) Toeplitz matrix arises in many applications, such as time series analysis and digital signal processing. The $N \times N$ symmetric Toeplitz system $T_N\mathbf{x} = \mathbf{b}$ is conventionally solved by algorithms based on the Levinson recursion formula [10], [16] with $O(N^2)$ operations. Superfast algorithms with $O(N \log^2 N)$ complexity have been studied intensively in the last ten years [1], [2], [3], [13]. More recently, Strang [19] proposed using an iterative method, i.e., the preconditioned conjugate gradient (PCG) method, to solve SPD Toeplitz systems and, as a consequence, the design of effective Toeplitz preconditioners has received much attention.

Strang's preconditioner S_N [19] is obtained by preserving the central half-diagonals of T_N and using them to form a circulant matrix. Since S_N is circulant, the matrix-vector product $S_N^{-1}\mathbf{v}$ can be conveniently computed via fast Fourier transform (FFT) with $O(N \log N)$ operations. It has been shown by R. Chan and Strang [5], [7] that if T_N is generated by a positive function in the Wiener class, the eigenvalues of the preconditioned matrices $P_N^{-1}T_N$ are clustered between $(1-\epsilon, 1+\epsilon)$ except for a fixed number independent of N. Another preconditioner C_N was proposed by T. Chan [8] and is defined to be the circulant matrix that minimizes the Frobenius norm $||R_N - T_N||_F$ over all circulant matrices R_N of size $N \times N$. This turns out to be a simple optimization problem, and the elements of C_N can be computed directly from the elements of T_N. The spectrum of $C_N^{-1}T_N$ is asymptotically equivalent to that of $S_N^{-1}T_N$ [6], and thus C_N and S_N have similar asymptotic behavior. In addition to preconditioners in circulant matrix form, preconditioners in skew-circulant matrix form [9] have been studied by Huckle [14]. We recently proposed a general approach

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for constructing Toeplitz preconditioners [15]. Under this framework, preconditioners in circulant and skew-circulant matrix forms can be viewed as special cases and, more interestingly, preconditioners that are neither circulant nor skew-circulant can also be derived.

In [15], four new preconditioners $K_{i,N}$, $i = 1, 2, 3, 4$ were constructed, and it was demonstrated numerically that they have better convergence performances than other preconditioners for rational Toeplitz matrices. It was also observed in [15] that for T_N generated by a positive rational function of order (p, q) in the Wiener class, the spectra of the preconditioned matrix $P_N^{-1}T_N$ with preconditioners S_N and $K_{i,N}$, $i = 1, 2, 3, 4$, have strong regularities. These regularities are stated as follows. Let the eigenvalues of $P_N^{-1}T_N$ be classified into two classes, i.e., the outliers and the clustered eigenvalues, depending on whether they converge to 1 asymptotically. Then, (1) the number of outliers is at most $2\max(p, q)$; and (2) the clustered eigenvalues are confined in an interval $(1 - \epsilon, 1 + \epsilon)$ with the radius ϵ proportional to the magnitude of the last element in the generating sequence used to construct the preconditioner. The main objective of this research is to prove these two spectral properties analytically.

With the above spectral regularities, the number of iterations required to reduce the norm of the residual $||\mathbf{b} - T_N\mathbf{x}_k||$ by a constant factor does not increase with the problem size N so that the solution of the system $T_N\mathbf{x} = \mathbf{b}$ can be accomplished with $\max(p, q) \times O(N \log N)$ operations. In addition, the superior performance of preconditioners $K_{i,N}$ can be easily explained by these spectral regularities. That is, for T_N generated by a positive rational function in the Wiener class, the last elements used to construct $K_{i,N}$ and S_N are, respectively, t_N and $t_{\lceil N/2\rceil}$ so that the corresponding radii are $\epsilon_K = O(|t_N|)$ and $\epsilon_S = O(|t_{\lceil N/2\rceil}|)$. Since $O(|t_N|) << O(|t_{\lceil N/2\rceil}|)$ for sufficiently large N, the PCG method with preconditioners $K_{i,N}$ converges faster than with preconditioner S_N.

We should point out that the first spectral property was recently proved by Trefethen. In [23], he used the theory of CF (Carathéodory and Fejér) approximation [22] to show that $S_N^{-1}T_N$ has at most $1 + 2\max(p, q)$ distinct eigenvalues asymptotically. A different approach is adopted in this paper to prove this property for both $S_N^{-1}T_N$ and $K_{i,N}^{-1}T_N$ (see Lemmas 2 and 8). Besides, since the first property only characterizes the spectrum of $P_N^{-1}T_N$ for infinite N, whereas the second property characterizes the spectrum of $P_N^{-1}T_N$ for both finite and infinite N, our results have a greater generality.

There exist direct methods that solve rational Toeplitz systems with $\max(p, q) \times O(N)$ operations [11], [24], [25]. However, the PCG method has three advantages compared with these direct methods. First, to implement the PCG algorithm, we only need a finite segment of the generating sequence t_n, $n = 0, 1, \ldots, N - 1$, which is provided by the problem, rather than the precise formula of the rational generating function. Second, the PCG method can be easily parallelized due to the parallelism provided by FFT, and it is possible to reduce the time complexity to $\max(p, q) \times O(\log N)$. In contrast, these direct methods are sequential algorithms, and the time complexity cannot be further reduced. Third, the PCG method is more widely applicable. For example, it can also be applied to Toeplitz systems with nonrational Toeplitz generating functions or those arising from the multidimensional space.

This paper is organized as follows. In §2, we briefly review the construction of preconditioners $K_{i,N}$ and summarize some of their spectral properties studied in [15]. In §§3 and 4, we prove the desired spectral properties of $K_{i,N}^{-1}T_N$ described above. The main idea is to transform the original generalized eigenvalue problem to an equivalent problem with nearly banded Toeplitz matrices. A similar approach is used to study the spectral properties of $S_N^{-1}T_N$, which is presented in §5. In §6, we use the analysis in §§3–5 to determine the analytical eigenvalue distributions of $K_{i,N}^{-1}T_N$ and $S_N^{-1}T_N$ for Toeplitz matrices with a geometric generating sequence.

2. Construction and spectral properties of Toeplitz preconditioners $K_{i,N}, i = 1,2,3,4$.

Let T_m be a sequence of $m \times m$ symmetric positive definite Toeplitz matrices with generating sequence t_n. Then,

$$T_N = \begin{bmatrix} t_0 & t_1 & \cdot & t_{N-2} & t_{N-1} \\ t_1 & t_0 & t_1 & \cdot & t_{N-2} \\ \cdot & t_1 & t_0 & \cdot & \cdot \\ t_{N-2} & \cdot & \cdot & \cdot & t_1 \\ t_{N-1} & t_{N-2} & \cdot & t_1 & t_0 \end{bmatrix}.$$

Preconditioners $K_{i,N}$, $i = 1,2,3,4$, for T_N are constructed by relating T_N to a $2N \times 2N$ circulant matrix R_{2N},

$$R_{2N} = \begin{bmatrix} T_N & \triangle T_N \\ \triangle T_N & T_N \end{bmatrix},$$

where $\triangle T_N$ is determined by the elements of T_N to make R_{2N} circulant, i.e.,

$$\triangle T_N = \begin{bmatrix} c & t_{N-1} & \cdot & t_2 & t_1 \\ t_{N-1} & c & t_{N-1} & \cdot & t_2 \\ \cdot & t_{N-1} & c & \cdot & \cdot \\ t_2 & \cdot & \cdot & \cdot & t_{N-1} \\ t_1 & t_2 & \cdot & t_{N-1} & c \end{bmatrix}, \tag{2.1}$$

with a constant c. If the behavior of the sequence t_n is known, we choose c to be t_N. Otherwise, $c = 0$.

Consider the following augmented circulant system:

$$\begin{bmatrix} T_N & \triangle T_N \\ \triangle T_N & T_N \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix}. \tag{2.2}$$

The solution of the above circulant system can be computed efficiently via FFT with $O(N \log N)$ operations. Since (2.2) is equivalent to

$$(T_N + \triangle T_N)\mathbf{x} = \mathbf{b},$$

this implies that $(T_N + \triangle T_N)^{-1}\mathbf{b}$ can be computed efficiently and that

$$K_{1,N} = T_N + \triangle T_N$$

can be used as a preconditioner for T_N. Three other preconditioners can be constructed in a similar way by assuming negative, even and odd periodicities for $\mathbf{x}$ and $\mathbf{b}$. We summarize the augmented systems and the corresponding preconditioners as follows:

$$\begin{bmatrix} T_N & \triangle T_N \\ \triangle T_N & T_N \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ -\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} \quad \text{and} \quad K_{2,N} = T_N - \triangle T_N,$$

$$\begin{bmatrix} T_N & \triangle T_N \\ \triangle T_N & T_N \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ J_N\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ J_N\mathbf{b} \end{bmatrix} \quad \text{and} \quad K_{3,N} = T_N + J_N\triangle T_N,$$

$$\begin{bmatrix} T_N & \triangle T_N \\ \triangle T_N & T_N \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ -J_N\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ -J_N\mathbf{b} \end{bmatrix} \quad \text{and} \quad K_{4,N} = T_N - J_N\triangle T_N,$$

where J_N is the $N \times N$ symmetric elementary matrix which has, by definition, ones along the secondary diagonal and zeros elsewhere ($J_{N,i,j} = 1$ if $i + j = N + 1$ and $J_{N,i,j} = 0$ if $i + j \neq N + 1$).

Since preconditioners $K_{i,N}$, $i = 1, 2, 3, 4$, correspond to $2N$-circulant systems, the matrix-vector product $K_{i,N}^{-1}\mathbf{v}$ for an arbitrary $\mathbf{v}$ can be achieved via $2N$-point FFT with $O(N \log N)$ operations. However, we should point out that $K_{1,N}$ is circulant and $K_{2,N}$ is skew-circulant so that $K_{1,N}^{-1}\mathbf{v}$ and $K_{2,N}^{-1}\mathbf{v}$ can be computed via N-point FFT. Although preconditioners $K_{3,N}$ and $K_{4,N}$ are neither circulant nor skew-circulant, $K_{3,N}^{-1}\mathbf{v}$ and $K_{4,N}^{-1}\mathbf{v}$ can be computed via N-point fast cosine and sine transforms, respectively. The operation count for N-point fast cosine (or sine) transform is approximately equal to that of N-point FFT in both the order and the proportional constant [17], [18], [27]. Therefore, the computational cost for the preconditioning step $K_{i,N}^{-1}\mathbf{v}$ with $i = 1, 2, 3, 4$ is about the same. For more details in implementing the PCG algorithm, we refer to [15].

To understand the relationship between the spectra of $K_{i,N}^{-1}T_N$, $i = 1, 2, 3, 4$, we rewrite the eigenvalues of $K_{i,N}^{-1}T_N$ as

$$\begin{aligned} [\lambda(K_{i,N}^{-1}T_N)]^{-1} = \lambda(T_N^{-1}(T_N + K_{i,N} - T_N)) &= \lambda(I + T_N^{-1}(K_{i,N} - T_N)) \\ &= 1 + \lambda(T_N^{-1}(K_{i,N} - T_N)), \end{aligned} \tag{2.3}$$

and examine the relationship between the spectra of $T_N^{-1}(K_{i,N} - T_N)$. This is characterized by the following theorem.

THEOREM 1. *Let Q_i be the set of the absolute values of the eigenvalues of $T_N^{-1}(K_{i,N} - T_N)$, i.e.,*

$$Q_i = \{|\lambda| : (K_{i,N} - T_N)\mathbf{x} = \lambda T_N \mathbf{x}\}, \qquad i = 1, 2, 3, 4.$$

Then, $Q_1 = Q_2 = Q_3 = Q_4$.

Proof. See [15] for the proof. □

The above theorem can be stated alternatively as follows. Let λ be an arbitrary eigenvalue of $T_N^{-1}(K_{i,N} - T_N)$; then there exists an eigenvalue of $T_N^{-1}(K_{j,N} - T_N)$, where $j \neq i$, with magnitude $|\lambda|$. From (2.3), spectra of $T_N^{-1}(K_{i,N} - T_N)$ clustered around zero are equivalent to those of $K_{i,N}^{-1}T_N$ clustered around unity. Since spectra of $T_N^{-1}(K_{i,N} - T_N)$ are clustered in a very similar pattern, so are those of $K_{i,N}^{-1}T_N$.

We assume that the generating sequence t_n for the sequence of Toeplitz matrices T_m satisfies the following two conditions:

$$\sum_{-\infty}^{\infty} |t_n| < \infty, \tag{2.4}$$

$$T(e^{i\theta}) = \sum_{-\infty}^{\infty} t_n e^{-in\theta} \geq \delta > 0 \quad \forall \theta, \tag{2.5}$$

and the resulting matrices are said to be generated by a positive function in the Wiener class. Since $T(e^{i\theta})$ describes the asymptotic eigenvalue distribution of T_m, the above conditions assume that the eigenvalues of T_m are bounded and uniformly positive, asymptotically. With (2.4) and (2.5), two spectral properties of $K_{i,N}^{-1}T_N$ are derived.

THEOREM 2. *Preconditioners $K_{i,N}$, $i = 1,2,3,4$, for symmetric positive definite Toeplitz matrices T_N with the generating sequence satisfying* (2.4) *and* (2.5) *are uniformly positive definite and bounded for sufficiently large N.*

Proof. See [15] for the proof. □

THEOREM 3. *Let T_N be the $N \times N$ matrix in a sequence of $m \times m$ symmetric positive definite Toeplitz matrices T_m with the generating sequence satisfying* (2.4) *and* (2.5). *The eigenvalues of the matrix $T_N^{-1}(K_{i,N} - T_N)$ are clustered between $(-\epsilon, +\epsilon)$ except for a finite number of outliers for sufficiently large $N(\epsilon)$.*

Proof. See [15] for the proof. □

Theorems 2 and 3 hold for both rational and nonrational Toeplitz matrices satisfying (2.4) and (2.5). However, when T_N is additionally rational, we are able to obtain stronger results and characterize the spectra of $K_{i,N}^{-1}T_N$ more precisely. In §§3 and 4, we focus on the spectrum of $K_{1,N}^{-1}T_N$, from which the spectra of $K_{i,N}^{-1}T_N$, $i = 2,3,4$, can be estimated based on Theorem 1.

3. Rational generating functions for $\triangle T_N$.

Due to (2.3), the spectral properties of $K_{1,N}^{-1}T_N$ can be determined by examining those of $T_N^{-1}\triangle T_N$, where $\triangle T_N$ is given in (2.1) with $c = t_N$. Let t_n, $-\infty < n < \infty$, be the generating sequence of a sequence of $m \times m$ Toeplitz matrices T_m. The Laurent series

$$T(z) = \sum_{n=-\infty}^{\infty} t_n z^{-n}$$

is known as the generating function of these matrices. If matrices T_m are symmetric, we decompose $T(z)$ into

$$T(z) = T_+(z^{-1}) + T_+(z), \tag{3.1}$$

where

$$T_+(z^{-1}) = \frac{t_0}{2} + \sum_{n=1}^{\infty} t_n z^{-n}. \tag{3.2}$$

Thus $T(z)$ is completely characterized by $T_+(z^{-1})$. Additionally, if

$$T_+(z^{-1}) = \frac{A(z^{-1})}{B(z^{-1})} = \frac{a_0 + a_1 z^{-1} + \cdots + a_p z^{-p}}{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}, \tag{3.3}$$

where $b_0 = 1$, $a_p b_q \neq 0$, and polynomials $A(z^{-1})$ and $B(z^{-1})$ have no common factor, we call T_m the rational Toeplitz matrices generated by a rational function of order (p,q). From (3.1) and (3.3), we have

$$T(z) = \frac{A(z^{-1})}{B(z^{-1})} + \frac{A(z)}{B(z)}. \tag{3.4}$$

It is well known [12] that there exists an isomorphism between the ring of the power series $P(z^{-1}) = \sum_{n=0}^{\infty} p_n z^{-n}$ (or $P(z)$) and the ring of the semi-infinite lower (or upper) triangular Toeplitz matrices with $p_0, p_1, \cdots, p_n, \cdots$ as the first column (or row). The power series multiplication is isomorphic to matrix multiplication. By applying the isomorphism to (3.4) and focusing on the leading $N \times N$ blocks of the corresponding matrices, we derive the following relationship [12]:

$$T_N = L_a L_b^{-1} + U_a U_b^{-1}, \tag{3.5}$$

where L_a (or U_a) is an $N \times N$ lower (or upper) triangular Toeplitz matrix with first N coefficients in $A(z^{-1})$ as its first column (or row). Matrices L_b and U_b are defined similarly with respect to $B(z^{-1})$. We can also establish an expression similar to (3.5) for $\triangle T_N$. To do so, we first note that the sequence t_n is recursively defined for large n. This is stated as follows.

LEMMA 1. *The sequence t_n generated by* (3.2) *and* (3.3) *follows the recursion,*

$$t_{n+1} = -(b_1 t_n + b_2 t_{n-1} + \cdots + b_q t_{n-q+1}), \qquad n \geq \max(p,q). \tag{3.6}$$

Proof. From (3.2) and (3.3), we have

$$\left(\frac{t_0}{2} + \sum_{n=1}^{\infty} t_n z^{-n}\right)(b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}) = a_0 + a_1 z^{-1} + \cdots + a_p z^{-p}.$$

The proof is completed by comparing the coefficients of the above equation. □

With Lemma 1, the number of outliers of $T_N^{-1}\triangle T_N$ is determined by the following lemma.

LEMMA 2. *Let T_N be an $N \times N$ symmetric Toeplitz matrix generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and the corresponding generating sequence satisfies* (2.4) *and* (2.5). *$T_N^{-1}\triangle T_N$ has asymptotically at most $2\max(p,q)$ nonzero eigenvalues* (*outliers*).

Proof. Let us define a matrix

$$\triangle E_N = \triangle F_N + \triangle F_N^T,$$

where

$$\triangle F_N = \begin{bmatrix} t_N & t_{N-1} & \cdot & t_2 & t_1 \\ t_{N+1} & t_N & t_{N-1} & \cdot & t_2 \\ \cdot & t_{N+1} & t_N & \cdot & \cdot \\ t_{2N-2} & \cdot & \cdot & \cdot & t_{N-1} \\ t_{2N-1} & t_{2N-2} & \cdot & t_{N+1} & t_N \end{bmatrix}.$$

Since elements t_n in $\triangle F_N$ satisfy (3.6), there are at most $\max(p,q)$ independent rows in $\triangle F_N$ and therefore, the rank of $\triangle E_N$ is at most $2\max(p,q)$.

Let $\triangle P_N = \triangle E_N - \triangle T_N$; it is easy to verify that the l_1 and l_∞ norms of $\triangle P_N$ are both less than

$$\tau_K = 2\sum_{n=N}^{2N-1} |t_n|.$$

Consequently, we have

$$||\triangle P_N||_2 \leq (||\triangle P_N||_1 ||\triangle P_N||_\infty)^{1/2} \leq \tau_K.$$

Since τ_K goes to zero as N goes to infinity due to (2.4), and since the eigenvalues of T_N^{-1} are bounded due to (2.5), the spectra of $T_N^{-1}\triangle T_N$ and $T_N^{-1}\triangle E_N$ are asymptotically equivalent. It follows that both $T_N^{-1}\triangle E_N$ and $T_N^{-1}\triangle T_N$ have at most $2\max(p,q)$ nonzero eigenvalues asymptotically. □

As a consequence of Lemma 2, $T_N^{-1}\triangle T_N$ has at least $N - 2\max(p,q)$ eigenvalues converging to zero as the problem size N becomes large. For the rest of this section and in §4, we study the clustering property of these eigenvalues. Our approach is outlined as follows. First, we associate $\triangle T_N$ with some appropriate rational generating function $\tilde{T}(z) = \tilde{T}_+(z^{-1}) + \tilde{T}_+(z)$. The forms of $\tilde{T}_+(z^{-1})$ for $p \le q$ and $p > q$ are given in Lemmas 3 and 4, respectively. We then transform the generalized eigenvalue problem involving $T_N^{-1}\triangle T_N$ into another generalized eigenvalue problem involving $Q_N^{-1}\triangle Q_N$. We show that Q_N and $\triangle Q_N$ are nearly banded Toeplitz matrices in Lemma 5 and examine the spectral property of $Q_N^{-1}\triangle Q_N$ in Lemma 6.

Since T_N is a symmetric rational Toeplitz matrix, and the elements of $\triangle T_N$ are those of T_N with reverse ordering, it is not surprising that $\triangle T_N$ is also generated by a certain rational function, which is determined below. Let us use the elements t_n of a given T_N with $N > \max(p, q)$ to construct a new sequence $\tilde{t}_n$. The cases $p \le q$ and $p > q$ are considered separately.

Case 1. $p \le q$. We choose

$$\tilde{t}_n = \begin{cases} t_{N-n}, & 0 \le n \le q-1, \\ -(\sum_{k=1}^{q} b_{q-k}\tilde{t}_{n-k})/b_q, & q \le n. \end{cases} \tag{3.7}$$

Note that elements $\tilde{t}_n$ above with $n \ge q$ are obtained based on the recursion (3.6) examined from the reverse direction.

Case 2. $p > q$. We decompose $T_+(z^{-1})$ into

$$T_+(z^{-1}) = F_+(z^{-1}) + T_{1,+}(z^{-1}), \tag{3.8}$$

where

$$F_+(z^{-1}) = f_0 + f_1 z^{-1} + \cdots + f_{p-q} z^{-(p-q)} \tag{3.8a}$$

and

$$T_{1,+}(z^{-1}) = \frac{A'(z^{-1})}{B(z^{-1})} = \frac{a'_0 + a'_1 z^{-1} + \cdots + a'_s z^{-s}}{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}, \tag{3.8b}$$

with $s < q$. Let $t_{1,n}$ be the generating sequence of $T_{1,+}(z^{-1})$. There exists a simple relationship between the elements of generating sequences for $T_+(z^{-1})$ and $T_{1,+}(z^{-1})$, i.e.,

$$t_n = \begin{cases} t_{1,n} + f_n, & 0 \le n \le p-q, \\ t_{1,n}, & p-q < n. \end{cases}$$

With respect to $T_{1,+}(z^{-1})$ and $F_+(z^{-1})$, we choose the corresponding $\tilde{t}_{1,n}$ and $\tilde{t}_{2,n}$, respectively, as

$$\tilde{t}_{1,n} = \begin{cases} t_{1,N-n}, & 0 \le n \le q-1, \\ -(\sum_{k=1}^{q} b_{q-k}\tilde{t}_{1,n-k})/b_q, & q \le n, \end{cases}$$

and

$$\tilde{t}_{2,n} = \begin{cases} f_{N-n}, & N-p+q \le n \le N, \\ 0, & \text{elsewhere}. \end{cases}$$

Finally, we define

$$\tilde{t}_n = \tilde{t}_{1,n} + \tilde{t}_{2,n}. \tag{3.9}$$

We associate the sequence $\tilde{t}_n$ given by (3.7) or (3.9) with a sequence of symmetric Toeplitz matrices $\tilde{T}_m$. It is straightforward to verify that for $N > \max(p,q)$, $\tilde{T}_N = \triangle T_N$. The generating function for matrices $\tilde{T}_m$ is

$$\tilde{T}(z) = \tilde{T}_+(z^{-1}) + \tilde{T}_+(z), \quad \text{where } \tilde{T}_+(z^{-1}) = \frac{\tilde{t}_0}{2} + \sum_{n=1}^{\infty} \tilde{t}_n z^{-n}.$$

The forms of $\tilde{T}_+(z^{-1})$ with $p \leq q$ and $p > q$ are described, respectively, in Lemmas 3 and 4.

LEMMA 3. *If T_N is generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and $p \leq q < N$, then $\triangle T_N$ is generated by $\tilde{T}(z)$ with*

$$\tilde{T}_+(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})} = \frac{c_0 + c_1 z^{-1} + \cdots + c_q z^{-q}}{d_0 + d_1 z^{-1} + \cdots + d_q z^{-q}}, \tag{3.10}$$

where

$$\text{(3.10a)} \quad d_i = \begin{cases} b_q^{-1} b_{q-i}, & 0 \leq i \leq q, \\ 0, & q < i, \end{cases} \qquad c_i = \begin{cases} \sum_{j=0}^{i} d_j \tilde{t}'_{i-j}, & 0 \leq i \leq q, \\ 0, & q < i, \end{cases}$$

and where

$$\text{(3.10b)} \qquad \tilde{t}'_n = \begin{cases} \tilde{t}_n, & n \neq 0, \\ \frac{\tilde{t}_0}{2}, & n = 0, \end{cases}$$

with $\tilde{t}_n$ given by (3.7).

Proof. By (3.7), the sequence $\tilde{t}_n$ satisfies the recursion

$$\tilde{t}_k = -(d_1 \tilde{t}_{k-1} + d_2 \tilde{t}_{k-2} + \cdots + d_q \tilde{t}_{k-q}), \quad \text{for } k \geq q, \tag{3.11}$$

with d_i given by (3.10a). Let us define $G_k(z^{-k})$, $k > q$, as

$$G_k(z^{-k}) = (\tilde{t}_k + d_1 \tilde{t}_{k-1} + d_2 \tilde{t}_{k-2} + \cdots + d_q \tilde{t}_{k-q}) z^{-k}.$$

It is evident from (3.11) that $G_k(z^{-k}) = 0$ for $k > q$. Therefore, we have

$$\begin{aligned}(1 + d_1 z^{-1} + \cdots + d_q z^{-q}) \tilde{T}_+(z^{-1}) &= \sum_{i=0}^{q} \left(\sum_{j=0}^{i} d_j \tilde{t}'_{i-j} \right) z^{-i} + \sum_{k=q+1}^{\infty} G_k(z^{-k}) \\ &= c_0 + c_1 z^{-1} + \cdots + c_q z^{-q},\end{aligned}$$

with c_i and $\tilde{t}'_i$ defined in (3.10a) and (3.10b), respectively. This completes the proof. □

LEMMA 4. *If T_N is generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and $q < p < N$, then $\triangle T_N$ is generated by $\tilde{T}(z)$ with*

$$\tilde{T}_+(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})} + F_+(z) z^{-N}, \tag{3.12}$$

where

$$\text{(3.12a)} \quad d_i = \begin{cases} b_q^{-1} b_{q-i}, & 0 \leq i \leq q, \\ 0, & q < i, \end{cases} \qquad c_i = \begin{cases} \sum_{j=0}^{i} d_j \tilde{t}'_{i-j}, & 0 \leq i \leq q, \\ 0, & q < i, \end{cases}$$

and where

$$\tilde{t}'_n = \begin{cases} \tilde{t}_n, & n \neq 0, \\ \frac{\tilde{t}_0}{2}, & n = 0, \end{cases} \tag{3.12b}$$

with $\tilde{t}_n$ *given by* (3.9).

Proof. Due to (3.8), we express $\tilde{T}_+(z^{-1})$ as $\tilde{T}_+(z^{-1}) = \tilde{F}_+(z^{-1}) + \tilde{T}_{1,+}(z^{-1})$, where

$$\tilde{F}_+(z^{-1}) = \sum_{n=N-p+q}^{N} f_{N-n} z^{-n}, \qquad \tilde{T}_{1,+}(z^{-1}) = \frac{\tilde{t}_{1,0}}{2} + \sum_{n=1}^{\infty} \tilde{t}_{1,n} z^{-n}.$$

It is clear from Lemma 3 and (3.8a) that

$$\tilde{F}_+(z^{-1}) = F_+(z) z^{-N}, \qquad \tilde{T}_{1,+}(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})}.$$

Thus the proof is completed. □

We rewrite (3.12) as

$$\tilde{T}_+(z^{-1}) = \frac{C_1(z^{-1})}{D(z^{-1})}, \quad \text{with } C_1(z^{-1}) = C(z^{-1}) + D(z^{-1}) F_+(z) z^{-N}. \tag{3.13}$$

Applying the isomorphism to (3.10) or (3.13) and focusing on the leading $N \times N$ blocks of the corresponding matrices, we obtain

$$\tilde{T}_N = L_c L_d^{-1} + U_c U_d^{-1},$$

where L_c (or U_c) is an $N \times N$ lower (or upper) triangular Toeplitz matrix with the first N coefficients of $C(z^{-1})$ ($p \leq q$) or $C_1(z^{-1})$ ($p > q$) as its first column (or row). Matrices L_d and U_d are similarly defined with respect to $D(z^{-1})$. Since $\triangle T_N = \tilde{T}_N$, we obtain

$$\triangle T_N = L_c L_d^{-1} + U_c U_d^{-1}. \tag{3.14}$$

4. Spectral properties of $T_N^{-1} \triangle T_N$. With the results given by (3.5) and (3.14), we then transform the generalized eigenvalue problem,

$$\triangle T_N \mathbf{x} = \lambda T_N \mathbf{x}, \tag{4.1}$$

to an equivalent generalized eigenvalue problem,

$$\triangle Q_N \mathbf{y} = \lambda Q_N \mathbf{y}, \tag{4.2}$$

where

$$Q_N = L_b T_N U_b = L_a U_b + L_b U_a \tag{4.2a}$$

and

$$\triangle Q_N = L_b \triangle T_N U_b = L_b L_c L_d^{-1} U_b + L_b U_c U_d^{-1} U_b. \tag{4.2b}$$

It is clear that (4.1) and (4.2) have identical eigenvalues and their eigenvectors are related via $\mathbf{x} = U_b \mathbf{y}$. The reason for (4.2) is that Q_N and $\triangle Q_N$ are nearly banded

Toeplitz matrices which can be more easily analyzed. The properties of matrices Q_N and $\triangle Q_N$ are characterized below.

LEMMA 5. *Let T_m be a sequence of $m \times m$ symmetric Toeplitz matrices generated by $T(z)$ with $T_+(z^{-1})$ given by (3.3), and the corresponding generating sequence satisfies (2.4) and (2.5). The southeast $(N - \max(p,q)) \times (N - \max(p,q))$ blocks of Q_N and $\triangle Q_N$ are symmetric banded Toeplitz matrices with generating functions*

$$Q(z) = A(z^{-1})B(z) + B(z^{-1})A(z) \tag{4.3}$$

and

$$\triangle Q(z) = b_q z^q C(z^{-1})B(z^{-1}) + b_q z^{-q} C(z)B(z), \tag{4.4}$$

respectively.

Proof. Consider two Toeplitz matrices F_N and G_N of size $N \times N$, where F_N is a lower triangular Toeplitz matrix with lower bandwidth r and the generating function $F(z^{-1})$, G_N an upper triangular Toeplitz matrix with upper bandwidth s and the generating function $G(z)$. It is easy to verify that the product $F_N G_N$, except for its northwest $r \times s$ block, is a banded Toeplitz matrix with the lower bandwidth r, upper bandwidth s, and generating function $F(z^{-1})G(z)$. We generalize the above result to $Q_N = L_a U_b + L_b U_a$ and find that the southeast $(N - \max(p,q)) \times (N - \max(p,q))$ block of Q_N is a symmetric banded Toeplitz matrix with the generating function

$$Q(z) = A(z^{-1})B(z) + B(z^{-1})A(z).$$

Since the product of lower (or upper) triangular Toeplitz matrices is commutative, we rewrite (4.2b) as

$$\triangle Q_N = \triangle Q_{1,N} + \triangle Q_{1,N}^T, \quad \text{where } \triangle Q_{1,N} = L_b L_c L_d^{-1} U_b.$$

When $p \le q$, the product $L_b L_c L_d^{-1}$ results in a lower triangular Toeplitz matrix with the generating function $B(z^{-1})C(z^{-1})D^{-1}(z^{-1})$. The matrix $\triangle Q_{1,N}$, except for the first q columns, is a Toeplitz matrix with the generating function

$$\triangle Q_{1,N}(z^{-1}) = B(z^{-1})C(z^{-1})D^{-1}(z^{-1})B(z).$$

We use (3.10a) to relate $D(z^{-1})$ with $B(z)$, i.e.,

$$D(z^{-1}) = \sum_{n=0}^{q} d_n z^{-n} = b_q^{-1} z^{-q} \sum_{n=0}^{q} b_{q-n} z^{q-n} = b_q^{-1} z^{-q} B(z).$$

Thus $\triangle Q_{1,N}(z^{-1}) = b_q z^q B(z^{-1})C(z^{-1})$. Similarly, $\triangle Q_{1,N}^T$, except for the first q rows, is a Toeplitz matrix with the generating function $\triangle Q_{1,N}(z)$. Therefore, the southeast $(N-q) \times (N-q)$ block of $\triangle Q_N$ is a symmetric banded Toeplitz matrix with the generating function

$$\triangle Q(z) = \triangle Q_{1,N}(z^{-1}) + \triangle Q_{1,N}(z) = b_q \ (z^q B(z^{-1})C(z^{-1}) + z^{-q} B(z)C(z)),$$

where the coefficients of $C(z^{-1})$ are given in Lemma 3.

When $p > q$, the generating function of matrix L_c is $C_1(z^{-1})$ in (3.13). Consequently, $\triangle Q_{1,N}$, except for the first q columns, is a Toeplitz matrix with the generating function

$$\begin{aligned}\triangle Q_{1,N}(z^{-1}) &= B(z^{-1})C_1(z^{-1})D^{-1}(z^{-1})B(z)\\ &= B(z^{-1})C(z^{-1})D^{-1}(z^{-1})B(z) + z^{-N}B(z^{-1})F_+(z)B(z).\end{aligned}$$

Recall that the orders of polynomials $B(z)$ and $F_+(z)$ are q and $p-q$, respectively. The lowest order in z of the polynomial $z^{-N}B(z^{-1})F_+(z)B(z)$ is $-(N-p)$, and the elements of the leading $N\times N$ Toeplitz matrix generated by $z^{-N}B(z^{-1})F_+(z)B(z)$ are zeros except for the southwest p diagonals. Therefore, the matrix $\triangle Q_{1,N}$, except for the first q columns and the southwest p diagonals, is a Toeplitz matrix with the generating function

$$\triangle Q_{1,N}(z^{-1}) = B(z^{-1})C(z^{-1})D^{-1}(z^{-1})B(z).$$

Then it follows that the southeast $(N-p)\times(N-p)$ block of $\triangle Q_N$ is a symmetric banded Toeplitz matrix with the generating function

$$\triangle Q(z) = b_q\ [z^q B(z^{-1})C(z^{-1}) + z^{-q}B(z)C(z)],$$

where the coefficients of $C(z^{-1})$ are given in Lemma 4. The proof is completed. □

The following lemma gives the bound of the clustered eigenvalues of $Q_N^{-1}\triangle Q_N$.

LEMMA 6. *Let T_m be a sequence of $m\times m$ symmetric Toeplitz matrices generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and let the corresponding generating sequence satisfy* (2.4) *and* (2.5). *Then $Q_N^{-1}\triangle Q_N$ has at least $N-2\max(p,q)$ eigenvalues with magnitude bounded by*

$$\epsilon = \max_{z=e^{-i2\pi n/N}} \left|\frac{\triangle Q(z)}{Q(z)}\right|. \tag{4.5}$$

Proof. Let us denote the southeast $(N-\max(p,q))\times(N-\max(p,q))$ blocks of Q_N and $\triangle Q_N$ by $\mathcal{Q}_{N-\max(p,q)}$ and $\triangle\mathcal{Q}_{N-\max(p,q)}$, respectively. By the minimax theorem (or *Courant–Fisher theorem*) of eigenvalues [20], [26], there are at least $N-2\max(p,q)$ eigenvalues of $Q_N^{-1}\triangle Q_N$ bounded by the maximum and the minimum eigenvalues of $\mathcal{Q}_{N-\max(p,q)}^{-1}\triangle\mathcal{Q}_{N-\max(p,q)}$.

It is clear from Lemma 5 that $\mathcal{Q}_{N-\max(p,q)}$ and $\triangle\mathcal{Q}_{N-\max(p,q)}$ are symmetric banded Toeplitz matrices with bandwidth $\le \max(p,q)$. We construct two $N\times N$ symmetric circulant matrices $\mathcal{R}_N$ and $\triangle\mathcal{R}_N$ with $\mathcal{Q}_{N-\max(p,q)}$ and $\triangle\mathcal{Q}_{N-\max(p,q)}$ as their leading principal submatrices, respectively. By the separation theorem (or intertwining theorem) of eigenvalues [20], [26], the eigenvalues of $\mathcal{Q}_{N-\max(p,q)}^{-1}\triangle\mathcal{Q}_{N-\max(p,q)}$ are bounded by the maximum and the minimum eigenvalues of $\mathcal{R}_N^{-1}\triangle\mathcal{R}_N$. It is well known that the eigenvalues of $\mathcal{R}_N^{-1}\triangle\mathcal{R}_N$ are

$$\triangle Q(e^{-i2\pi n/N})/Q(e^{-i2\pi n/N}), \qquad n=0,1,\cdots,N-1.$$

Thus the proof is completed. □

We then focus on the bound of (4.5). By using (3.1) and (3.3), $\triangle Q(z)/Q(z)$ can be further simplified as

$$\begin{aligned}\triangle Q(z)/Q(z) &= [b_q z^q B(z^{-1})C(z^{-1}) + b_q z^{-q}B(z)C(z)]/[B(z^{-1})B(z)T(z)]\\ &= [b_q z^q C(z^{-1})]/[B(z)T(z)] + [b_q z^{-q}C(z)]/[B(z^{-1})T(z)].\end{aligned} \tag{4.6}$$

Since $T(e^{i\theta}) = A(e^{-i\theta})/B(e^{-i\theta}) + A(e^{i\theta})/B(e^{i\theta})$, and $|T(e^{i\theta})|$ is finite from (2.4), $|B(e^{i\theta})|$ is uniformly positive, i.e.,

$$|B(e^{i\theta})| \ge \beta > 0. \tag{4.7}$$

Combining (2.5), (4.6), and (4.7), we obtain

$$\left|\frac{\triangle Q(e^{-i\theta})}{Q(e^{-i\theta})}\right| \le \left|\frac{2b_q C(e^{-i\theta})}{\beta\delta}\right|, \tag{4.8}$$

with arbitrary θ.

We then focus our discussion on the bound of $|b_q C(e^{-i\theta})|$. First, we have

$$|b_q C(e^{-i\theta})| \le \sum_{i=0}^{q} |b_q c_i| = \sum_{i=0}^{q} \left|\sum_{j=0}^{i} b_q d_j \tilde{t}'_{i-j}\right| = \sum_{i=0}^{q} \left|\sum_{j=0}^{i} b_{q-j} t_{N-i+j}\right|, \tag{4.9}$$

where the last equality is due to (3.7), (3.10a), and (3.10b). Since t_n satisfies the recursion (3.6), we use the equality

$$\sum_{j=0}^{q} b_{q-j} t_{N-i+j} = 0$$

with $N > \max(p,q)$ to simplify (4.9), i.e.,

$$|b_q C(e^{-i\theta})| \le \sum_{i=0}^{q} \left| - \sum_{j=i+1}^{q} b_{q-j} t_{N-i+j} \right| \le \sum_{i=0}^{q} \sum_{j=i+1}^{q} |b_{q-j}||t_{N-i+j}|$$

$$\le \max_{N \le n \le N+q} |t_n| \sum_{i=0}^{q} \sum_{j=i+1}^{q} |b_{q-j}|. \tag{4.10}$$

Furthermore, the term $\sum_{i=0}^{q} \sum_{j=i+1}^{q} |b_{q-j}|$ is bounded by

$$\sum_{i=0}^{q} \sum_{j=i+1}^{q} |b_{q-j}| < \sum_{i=0}^{q} \sum_{j=1}^{q} |b_{q-j}| < (q+1) \sum_{j=0}^{q} |b_j| < (q+1)2^q, \tag{4.11}$$

where the last inequality is due to the following lemma.

LEMMA 7. *Let T_m be a sequence of $m \times m$ symmetric Toeplitz matrices generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and let the corresponding generating sequence satisfy* (2.4) *and* (2.5); *then*

$$\sum_{j=0}^{q} |b_j| < 2^q.$$

Proof. Since $B(z^{-1})$ is a polynomial in z of order q, $B(z^{-1})$ can be factorized as

$$B(z^{-1}) = \sum_{i=0}^{q} b_i z^{-i} = (1 - r_1 z^{-1})(1 - r_2 z^{-1}) \cdots (1 - r_q z^{-1}), \tag{4.12}$$

where r_i, $1 \le i \le q$, are poles of $T_+(z^{-1})$. A direct consequence of (2.4) is that all poles of $T_+(z^{-1})$ should lie inside the unit circle, i.e., $|r_i| < 1$, $1 \le i \le q$. It is clear from (4.12) that

$$|b_k| \le \binom{q}{k} (\max |r_i|)^k < \binom{q}{k}, \qquad \text{where} \binom{q}{k} \equiv \frac{q!}{(q-k)!\,k!}.$$

Therefore, we have

$$\sum_{j=0}^{q} |b_j| < \sum_{j=0}^{q} \binom{q}{j} = 2^q,$$

and the proof is completed. □

Combining (4.8), (4.10), and (4.11), we have

$$\max_n |\triangle Q(e^{-i2\pi n/N})/Q(e^{-i2\pi n/N})| < \frac{2^{q+1}(q+1)}{\beta\delta} \max_{N \le n \le N+q} |t_n|.$$

Since $|r_i| < 1$, $1 \le i \le q$, t_n is monotonically decreasing and

$$\max_{N \le n \le N+q} |t_n| = |t_N|,$$

for sufficiently large N. Thus

$$\max_n |\triangle Q(e^{-i2\pi n/N})/Q(e^{-i2\pi n/N})| < \frac{2^{q+1}(q+1)|t_N|}{\beta\delta} = \epsilon_K. \tag{4.13}$$

By Lemma 6, there are at least $N - 2\max(p,q)$ eigenvalues of $Q_N^{-1}\triangle Q_N$ with magnitude bounded by ϵ_K in (4.13). Since eigenvalues of $T_N^{-1}\triangle T_N$ are equivalent to those of $Q_N^{-1}\triangle Q_N$, there are at least $N - 2\max(p,q)$ eigenvalues of $T_N^{-1}\triangle T_N$ with magnitude bounded by ϵ_K as well. When ϵ_K is small enough, there are at least $N-2\max(p,q)$ eigenvalues of $K_{i,N}^{-1}T_N$, $i = 1,2,3,4$, clustered between $(1-\epsilon_K, 1+\epsilon_K)$ for sufficiently large N. We summarize the analysis in this section into the following theorem.

THEOREM 4. *Let T_m be a sequence of $m \times m$ symmetric Toeplitz matrices generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and let the corresponding generating sequence satisfy* (2.4) *and* (2.5). *For sufficiently large N, the spectra of the preconditioned Toeplitz matrices $K_{i,N}^{-1}T_N$, $i = 1,2,3,4$, have the following two properties:*

P1. *The number of outliers is at most* $2\max(p,q)$.

P2. *There are at least $N-2\max(p,q)$ eigenvalues that lie between* $(1-\epsilon_K, 1+\epsilon_K)$, *where ϵ_K is given by* (4.13).

5. Discussion on Strang's preconditioners. We adopt a procedure similar to that described in §§3 and 4 to examine the spectral properties of $S_N^{-1}T_N$, where S_N is Strang's preconditioner. Only the cases where $p \le q$ and $N = 2M$ are discussed. Since the analysis for the cases where $p > q$ or N is odd can be performed in a straightforward way, it is omitted to avoid unnecessary repetition.

Recall that Strang's preconditioner S_N is obtained by preserving the central half-diagonals of T_N and using them to form a circulant matrix. That is, when $N = 2M$, S_N is defined as a symmetric Toeplitz matrix with the first row

$$S_N: \quad [t_0, t_1, \cdots, t_{M-1}, t_M, t_{M-1}, \cdots, t_1].$$

Let us denote the difference between S_N and T_N by $\triangle S_N$, i.e., $\triangle S_N = S_N - T_N$. The number of outliers of $S_N^{-1}T_N$ is determined by the following lemma.

LEMMA 8. *Let T_N be an $N \times N$ symmetric Toeplitz matrix generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and let the corresponding generating sequence satisfy* (2.4) *and* (2.5). *$T_N^{-1}\triangle S_N$ has asymptotically at most $2\max(p,q)$ nonzero eigenvalues* (*outliers*).

Proof. The proof is similar to that of Lemma 2. We use

$$\triangle E_N = \begin{bmatrix} 0 & \triangle F_M \\ \triangle F_M^T & 0 \end{bmatrix}$$

to approximate $\triangle S_N$, where

$$\triangle F_M = \begin{bmatrix} t_M & t_{M-1} & t_{M-2} & \cdots & t_3 & t_2 & t_1 \\ t_{M+1} & t_M & t_{M-1} & \cdots & \cdot & t_3 & t_2 \\ t_{M+2} & t_{M+1} & t_M & \cdots & \cdot & \cdot & t_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ t_{N-3} & \cdot & \cdot & \cdots & t_M & t_{M-1} & t_{M-2} \\ t_{N-2} & t_{N-3} & \cdot & \cdots & t_{M+1} & t_M & t_{M-1} \\ t_{N-1} & t_{N-2} & t_{N-3} & \cdots & t_{M+2} & t_{M+1} & t_M \end{bmatrix}.$$

Since elements t_n in $\triangle F_M$ satisfy the recursion described in Lemma 1, there are at most $\max(p,q)$ independent rows in $\triangle F_M$. Therefore, the rank of $\triangle E_N$ is at most $2\max(p,q)$. Let us define $\triangle P_N = \triangle E_N - \triangle S_N$. Then we find that

$$\triangle P_N = \begin{bmatrix} 0 & \triangle G_M \\ \triangle G_M^T & 0 \end{bmatrix},$$

where $\triangle G_M$ is an $M \times M$ symmetric Toeplitz matrix with the first row

$$\triangle G_M : \quad [t_M, t_{M+1}, t_{M+2}, \cdots, t_{N-3}, t_{N-2}, t_{N-1}].$$

It is easy to verify that, for sufficiently large N, the l_1 and l_∞ norms of $\triangle P_N$ are both less than

$$\tau_S = 2\sum_{n=M}^{N-1} |t_n|.$$

Consequently, we have

$$||\triangle P_N||_2 \le (||\triangle P_N||_1 ||\triangle P_N||_\infty)^{1/2} \le \tau_S.$$

Since τ_S goes to zero as M goes to infinity due to (2.4), and since the eigenvalues of T_N^{-1} are bounded due to (2.5), the spectra of $T_N^{-1}\triangle S_N$ and $T_N^{-1}\triangle E_N$ are asymptotically equivalent. It follows that both $T_N^{-1}\triangle E_N$ and $T_N^{-1}\triangle S_N$ have at most $2\max(p,q)$ nonzero eigenvalues, asymptotically. □

The matrix $\triangle S_N$ can be expressed as $\triangle S_N = \triangle S_{1,N} - \triangle S_{2,N}$, where

$$\triangle S_{1,N} = \begin{bmatrix} 0 & F_{1,M} \\ F_{1,M}^T & 0 \end{bmatrix} \quad \text{and} \quad \triangle S_{2,N} = \begin{bmatrix} 0 & F_{2,M} \\ F_{2,M}^T & 0 \end{bmatrix},$$

and where $F_{1,M}$ and $F_{2,M}$ are $M \times M$ upper triangular Toeplitz matrices with the following first rows:

$$F_{1,M} : \quad [t_M, t_{M-1}, t_{M-2}, \cdots, t_2, t_1],$$
$$F_{2,M} : \quad [t_M, t_{M+1}, t_{M+2}, \cdots, t_{N-2}, t_{N-1}].$$

We use t_n, which satisfies (3.6), to construct two new sequences:

$$\tilde{s}_{1,n} = \begin{cases} 0, & 0 \le n \le M-1, \\ t_{N-n}, & M \le n \le M+q-1, \\ -(\sum_{k=1}^{q} b_{q-k}\tilde{t}_{n-k})/b_q, & M+q \le n, \end{cases} \tag{5.1}$$
$$\tilde{s}_{2,n} = \begin{cases} 0, & 0 \le n \le M-1, \\ t_n, & M \le n, \end{cases}$$

and associate $\tilde{s}_{1,n}$ and $\tilde{s}_{2,n}$ with two sequences of symmetric Toeplitz matrices $\tilde{S}_{1,m}$ and $\tilde{S}_{2,m}$, $m = 1, 2, \cdots$, whose generating functions are defined as

$$\tilde{S}_1(z) = \tilde{S}_{1,+}(z^{-1}) + \tilde{S}_{1,+}(z), \quad \text{where} \quad \tilde{S}_{1,+}(z^{-1}) = \frac{\tilde{s}_{1,0}}{2} + \sum_{n=1}^{\infty} \tilde{s}_{1,n} z^{-n},$$

and

$$\tilde{S}_2(z) = \tilde{S}_{2,+}(z^{-1}) + \tilde{S}_{2,+}(z), \quad \text{where} \quad \tilde{S}_{2,+}(z^{-1}) = \frac{\tilde{s}_{2,0}}{2} + \sum_{n=1}^{\infty} \tilde{s}_{2,n} z^{-n},$$

respectively. We can easily verify that for $N > 2\max(p,q)$,

$$\tilde{S}_{1,N} = \triangle S_{1,N} \quad \text{and} \quad \tilde{S}_{2,N} = \triangle S_{2,N}.$$

Then, by using the same approach for proving Lemma 3, we obtain the following lemma.

LEMMA 9. *If T_N is generated by $T(z)$ with $T_+(z^{-1})$ given by (3.3), then*

$$\tilde{S}_{1,+}(z^{-1}) = \frac{z^{-M}\tilde{C}(z^{-1})}{D(z^{-1})} = \frac{\tilde{c}_0 + \tilde{c}_1 z^{-1} + \cdots + \tilde{c}_q z^{-q}}{d_0 + d_1 z^{-1} + \cdots + d_q z^{-q}} \tag{5.2a}$$

and

$$\tilde{S}_{2,+}(z^{-1}) = \frac{z^{-M}\tilde{A}(z^{-1})}{B(z^{-1})} = \frac{\tilde{a}_0 + \tilde{a}_1 z^{-1} + \cdots + \tilde{a}_q z^{-q}}{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}, \tag{5.2b}$$

where the coefficients b_i and d_i are given by (3.3) *and* (3.8), *and*

$$\tilde{a}_i = \begin{cases} \sum_{j=0}^{i} b_j t_{M+i-j}, & 0 \le i \le q, \\ 0, & q < i, \end{cases} \qquad \tilde{c}_i = \begin{cases} \sum_{j=0}^{i} d_j \tilde{s}_{1,M+i-j}, & 0 \le i \le q, \\ 0, & q < i, \end{cases}$$

with $\tilde{s}_{1,n}$ given by (5.1).

Thus $\triangle S_N$ can be decomposed into

$$\triangle S_N = \triangle S_{1,N} - \triangle S_{2,N} = L_{\tilde{c}} L_d^{-1} + U_{\tilde{c}} U_d^{-1} - L_{\tilde{a}} L_b^{-1} - U_{\tilde{a}} U_b^{-1}, \tag{5.3}$$

where $L_{\tilde{c}}$ (or $U_{\tilde{c}}$) is an $N \times N$ lower (or upper) triangular Toeplitz matrix with the first N coefficients of $z^{-M}\tilde{C}(z^{-1})$ as its first column (or row), and matrices $L_{\tilde{a}}$, L_b, and L_d (or $U_{\tilde{a}}$, U_b, and U_d) are similarly defined with respect to $z^{-M}\tilde{A}(z^{-1})$, $B(z^{-1})$, and $D(z^{-1})$, respectively.

By using the decomposition formulas (3.5) and (5.3), we transform the generalized eigenvalue problem

$$\triangle S_N \mathbf{x} = \lambda T_N \mathbf{x} \tag{5.4}$$

into another generalized eigenvalue problem

$$\triangle Q_{S,N}\mathbf{y} = \lambda Q_N \mathbf{y}, \tag{5.5}$$

where

$$\begin{aligned} Q_N = L_b T_N U_b &= L_a U_b + L_b U_a, \\ \triangle Q_{S,N} = L_b \triangle S_N U_b &= (L_b L_{\tilde{c}} L_d^{-1} U_b + L_b U_{\tilde{c}} U_d^{-1} U_b) - (L_{\tilde{a}} U_b + L_b U_{\tilde{a}}). \end{aligned}$$

The systems (5.4) and (5.5) have the same eigenvalues and their eigenvectors are related via $\mathbf{x} = U_b \mathbf{y}$. The matrix $\triangle Q_{S,N}$ is a nearly banded Toeplitz matrix characterized by the following lemma.

LEMMA 10. *Let T_m be a sequence of $m \times m$ symmetric Toeplitz matrices generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and the corresponding generating sequence satisfies* (2.4) *and* (2.5). *The southeast $(N - \max(p,q)) \times (N - \max(p,q))$ block of $\triangle Q_{S,N}$ is a symmetric banded Toeplitz matrix with the generating function*

$$\triangle Q_S(z) = B(z^{-1})\tilde{S}(z)B(z) = \triangle Q_{S,1}(z) - \triangle Q_{S,2}(z), \tag{5.6}$$

where

$$\begin{aligned} \triangle Q_{S,1}(z) &= b_q z^{-(M-q)} B(z^{-1})\tilde{C}(z^{-1}) + b_q z^{M-q} B(z)\tilde{C}(z), \\ \triangle Q_{S,2}(z) &= z^{-M}\tilde{A}(z^{-1})B(z) + z^M B(z^{-1})\tilde{A}(z). \end{aligned}$$

Since the generating sequence t_n of T_N satisfies conditions (2.4) and (2.5), we can use arguments given in the previous section and obtain

$$\left| \frac{\triangle Q_{S,1}(e^{-i\theta})}{Q(e^{-i\theta})} \right| \le \frac{2^{q+1}(q+1)|t_M|}{\beta\delta} = \epsilon'$$

and

$$\left| \frac{\triangle Q_{S,2}(e^{-i\theta})}{Q(e^{-i\theta})} \right| \le \frac{2^{q+1}(q+1)|t_M|}{\beta\delta} = \epsilon'$$

for arbitrary θ. By using arguments similar to those in Lemma 6, it can be derived that $T_N^{-1}\triangle S_N$ has at least $N - 2\max(p,q)$ eigenvalues bounded by

$$\epsilon_S = 2\epsilon' = \frac{2^{q+2}(q+1)|t_{N/2}|}{\beta\delta} \tag{5.7}$$

for sufficiently large N. The analysis in this section is concluded by the following theorem.

THEOREM 5. *Let T_m be a sequence of $m \times m$ symmetric Toeplitz matrices generated by $T(z)$ with $T_+(z^{-1})$ given by* (3.3), *and the corresponding generating sequence satisfies* (2.4) *and* (2.5). *For sufficiently large N, the spectrum of the preconditioned Toeplitz matrix $S_N^{-1}T_N$ has the following two properties:*

P1. *The number of outliers is at most $2\max(p,q)$.*

P2. *There are at least $N - 2\max(p,q)$ eigenvalues that lie between $(1-\epsilon_S, 1+\epsilon_S)$, where ϵ_S is given by* (5.7).

Let us compare the preconditioners $K_{i,N}$ and S_N. From Theorems 4 and 5, the spectra of $K_{i,N}^{-1}T_N$ and $S_N^{-1}T_N$ have the same number of outliers, and the other eigenvalues are clustered around 1 within radii ϵ_K and ϵ_S given by (4.13) and (5.7), respectively. It is clear that the parameters q, β, and δ are independent of the problem size N, and that the terms $|t_N|$ and $|t_{N/2}|$ determine the convergence rate of the PCG method. For sufficiently large N, we have $O(\epsilon_K) = O(\epsilon_S^2)$. This implies that, after the first several iterations which eliminate the effects of the outliers, the residual reduced by one iteration of the PCG method with preconditioners $K_{i,N}$ is about the same as that reduced by two iterations of the PCG method with preconditioner S_N. This has been confirmed by numerical experiments reported in [15].

6. The special case with geometric generating sequences. It has been observed from numerical experiments [15], [21], that the eigenvalues of $K_{1,N}^{-1}T_N$ and $S_N^{-1}T_N$ with T_N generated by the geometric sequence $t_n = t^n$, $|t| < 1$, are very regular. The observations are summarized as follows.

R1. The eigenvalues of $K_{1,N}^{-1}T_N$ are $(1+t)^{-1}$, $(1-t)^{-1}$, and $(1-t^N)^{-1}$ with multiplicities 1, 1, and $N-2$, respectively.

R2. When N is even ($N = 2M$), the eigenvalues of $S_N^{-1}T_N$ are $(1+t)^{-1}$, $(1-t)^{-1}$, 1, $(1+t^M)^{-1}$, and $(1-t^M)^{-1}$ with multiplicities 1, 1, 2, $M-2$, and $M-2$, respectively.

In this section, we provide an analytical approach to explain these two regularities.

First, we examine the preconditioner $K_{1,N}$. For the generating sequence $t_n = t^n$, its generating function is

$$T(z) = T_+(z^{-1}) + T_+(z), \quad \text{where } T_+(z^{-1}) = \frac{A(z^{-1})}{B(z^{-1})} = \frac{0.5 + 0.5tz^{-1}}{1 - tz^{-1}},$$

so that the order (p, q) of $T_+(z^{-1})$ is $(1, 1)$. From Lemma 3, we obtain

$$\tilde{T}_+(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})} = \frac{t^N(0.5 + 0.5t^{-1}z^{-1})}{1 - t^{-1}z^{-1}},$$

which is related to $\triangle T_N = K_{1,N} - T_N$. By using (4.3) and (4.4), we have

$$Q(z) = A(z^{-1})B(z) + B(z^{-1})A(z) = 1 - t^2 \tag{6.1}$$

and

$$\triangle Q(z) = -t[zB(z^{-1})C(z^{-1}) + z^{-1}B(z)C(z)] = -t^N(1 - t^2). \tag{6.2}$$

Note that $q = 1$ and $b_q = -t$ are used in deriving (6.2). Due to (6.1) and (6.2), the southeast $(N-1) \times (N-1)$ blocks of Q_N and $\triangle Q_N$ are identity matrices multiplied by the constants $1 - t^2$ and $-t^N(1-t^2)$, respectively. Consider the following linear combination of Q_N and $\triangle Q_N$:

$$V_N = \triangle Q_N + t^N Q_N.$$

It is clear that the southeast $(N-1) \times (N-1)$ block of V_N is a zero matrix. Since the first two columns are linearly independent, and any two columns of the last $N-1$ columns of V_N are linearly dependent, V_N has a null space of dimension $N-2$. This implies that $Q_N^{-1}\triangle Q_N$, or equivalently, $T_N^{-1}\triangle T_N$, has the eigenvalue $-t^N$ with multiplicity $N-2$. Therefore, $(T_N + \triangle T_N)^{-1}T_N = K_{1,N}^{-1}T_N$ has the eigenvalue $(1-t^N)^{-1}$ with multiplicity $N-2$.

To determine the remaining two eigenvalues, i.e., the outliers, we use the technique described in [4] to transform the problem $\triangle T_N \mathbf{x} = \lambda T_N \mathbf{x}$ to another equivalent problem. Consider the case with even N ($N = 2M$). Since $\triangle T_N$ and T_N are both symmetric Toeplitz matrices, they can be expressed in the following block matrix form:

$$\triangle T_N = \begin{bmatrix} \triangle T_{1,M} & \triangle T_{2,M}^T \\ \triangle T_{2,M} & \triangle T_{1,M} \end{bmatrix} \quad \text{and} \quad T_N = \begin{bmatrix} T_{1,M} & T_{2,M}^T \\ T_{2,M} & T_{1,M} \end{bmatrix}.$$

Let W_N be the orthonormal matrix

$$W_N = \frac{1}{\sqrt{2}} \begin{bmatrix} I_M & I_M \\ -J_M & J_M \end{bmatrix},$$

where I_M and J_M are $M \times M$ identity and symmetric elementary matrices, respectively. By using the transformation

$$W_N^{-1} \triangle T_N W_N \mathbf{y} = \lambda W_N^{-1} T_N W_N \mathbf{y},$$

we obtain two decoupled subproblems,

$$(\triangle T_{1,M} - J_M \triangle T_{2,M})\mathbf{y}_- = \lambda_-(T_{1,M} - J_M T_{2,M})\mathbf{y}_-, \tag{6.3}$$

$$(\triangle T_{1,M} + J_M \triangle T_{2,M})\mathbf{y}_+ = \lambda_+(T_{1,M} + J_M T_{2,M})\mathbf{y}_+, \tag{6.4}$$

where λ_- and λ_+ are also eigenvalues of the original problem $\triangle T_N \mathbf{x} = \lambda T_N \mathbf{x}$. Since the first rows of matrices on both sides of (6.3) are proportional by a constant $-t$, $\lambda_- = -t$ with $\mathbf{y}_- = \mathbf{e}_1$ (the unit vector with 1 at the first element) satisfies (6.3). Similarly, we can argue that $\lambda_+ = t$ with $\mathbf{y}_+ = \mathbf{e}_1$ is an eigenvalue-eigenvector pair for (6.4). Thus $1/(1-t)$ and $1/(1+t)$ are two outliers of $(T_N + \triangle T_N)^{-1} T_N = K_{1,N}^{-1} T_N$. When N is odd, the same result can be derived with a slightly modified W_N given in [4].

By using the relationship among preconditioners $K_{i,N}$, $i = 1, 2, 3, 4$, we can determine all eigenvalues of $K_{i,N}^{-1} T_N$. They all have three distinct eigenvalues (two outliers and $N-2$ clustered eigenvalues) summarized in Table 1.

TABLE 1
Eigenvalues of $K_{i,N}^{-1} T_N$.

	$K_{1,N}^{-1} T_N$	$K_{2,N}^{-1} T_N$	$K_{3,N}^{-1} T_N$	$K_{4,N}^{-1} T_N$
λ_1	$(1+t)^{-1}$	$(1+t)^{-1}$	$(1+t)^{-1}$	$(1-t)^{-1}$
λ_2	$(1-t)^{-1}$	$(1-t)^{-1}$	$(1+t^N)^{-1}$	$(1+t^N)^{-1}$
λ_3	$(1-t^N)^{-1}$	$(1+t^N)^{-1}$	$(1-t^N)^{-1}$	$(1-t^N)^{-1}$

Next, we examine Strang's preconditioner S_N with even N. When $N = 2M$, the two central rows of $S_N - T_N$ are zeros. This implies that $S_N^{-1} T_N$ has the eigenvalue 1 with multiplicity 2. By using (5.2a) and (5.2b), we have

$$\tilde{S}_{1,+}(z^{-1}) = \frac{z^{-M}\tilde{C}(z^{-1})}{D(z^{-1})} = \frac{z^{-M}t^M}{1 - t^{-1}z^{-1}},$$

$$\tilde{S}_{2,+}(z^{-1}) = \frac{z^{-M}\tilde{A}(z^{-1})}{B(z^{-1})} = \frac{z^{-M}t^M}{1 - tz^{-1}},$$

respectively. By substituting $\tilde{A}(z^{-1})$, $B(z^{-1})$, $\tilde{C}(z^{-1})$, and $D(z^{-1})$ into (5.6) and using (6.1), we obtain

$$\triangle Q_S(z) = -t^M(z^{-M} + z^M)(1 - t^2).$$

Then, the nonzero elements of $\triangle Q_{S,N-1}$, which is the southeast $(N-1)\times(N-1)$ block of $\triangle Q_{S,N}$, only occur along the $\pm M$th diagonals and take the same value $-t^M(1-t^2)$. Consider the linear combination of $\triangle Q_{S,N}$ and Q_N,

$$V_{1,N} = \triangle Q_{S,N} + t^M Q_N.$$

By adding the $k+1$th column to the $M+k+1$th column of $V_{1,N}$, for $k = 1, 2, \cdots (M-1)$, we find that the southeast $(N-1)\times(M-1)$ block of the resulting matrix is the zero matrix. Consequently, $V_{1,N}$ has a null space of dimension $M-2$ and $Q_N^{-1}\triangle Q_{S,N}$ has the eigenvalue $-t^M$ with multiplicity $M-2$. Similarly, we can show that

$$V_{2,N} = \triangle Q_{S,N} - t^M Q_N$$

has a null space of dimension $M-2$ by subtracting the $k+1$ column from the $M+k+1$th column of $V_{2,N}$, for $k = 1, 2, \cdots (M-1)$. Therefore, $Q_N^{-1}\triangle Q_{S,N}$ has the eigenvalue t^M with the same multiplicity $M-2$. As a consequence, $S_N^{-1}T_N$ has the eigenvalues $(1+t^M)^{-1}$ and $(1-t^M)^{-1}$ with multiplicity $M-2$.

To determine the remaining two eigenvalues of $S_N^{-1}T_N$, we use the same transformation discussed earlier and consider the eigenvalues of the following two subproblems:

$$(T_{1,M} - J_M T_{2,M})\mathbf{y}_- = \lambda_-(S_{1,M} - J_M S_{2,M})\mathbf{y}_-, \tag{6.5}$$

$$(T_{1,M} + J_M T_{2,M})\mathbf{y}_+ = \lambda_+(S_{1,M} + J_N S_{2,M})\mathbf{y}_+, \tag{6.6}$$

where $S_{1,M}$ and $S_{2,M}$ are the northwest and southwest $M \times M$ blocks of S_N, respectively. Since the first rows of matrices on both sides of (6.5) are proportional by a constant $1-t$, $\lambda_- = 1/(1-t)$ with $\mathbf{y}_- = \mathbf{e}_1$ satisfies (6.5). Similarly, $\lambda_+ = 1/(1+t)$ with $\mathbf{y}_+ = \mathbf{e}_1$ satisfies (6.6).

7. Conclusion. In this paper, we have proved the spectral properties of the preconditioned rational Toeplitz matrices $P_N^{-1}T_N$ with the preconditioner S_N proposed by Strang [19] and the preconditioners $K_{i,N}$ proposed by the authors [15]. The eigenvalues of $P_N^{-1}T_N$ are classified into two classes, i.e., the outliers and the clustered eigenvalues. The number of outliers depends on the order of the rational generating function. The clustered eigenvalues are confined in the interval $(1-\epsilon, 1+\epsilon)$ with the radii $\epsilon_K = O(|t_N|)$ and $\epsilon_S = O(|t_{N/2}|)$ for $K_{i,N}^{-1}T_N$ and $S_N^{-1}T_N$, respectively. When the symmetric Toeplitz matrix T_N is generated by the geometric sequence t^n with $|t| < 1$, the precise eigenvalue distributions of $K_{i,N}^{-1}T_N$ and $S_{2M}^{-1}T_{2M}$ have been determined analytically. Since the eigenvalues of $K_{i,N}^{-1}T_N$ are more closely clustered than those of $S_N^{-1}T_N$, preconditioners $K_{i,N}$ are more efficient for solving rational Toeplitz systems.

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