Performance Analysis of a Class of Multistage DS-CDMA Receivers for Multipath Channels

Sau-Hsuan Wu, Urbashi Mitra and C.-C. Jay Kuo
Department of Electrical Engineering
University of Southern California
Los Angeles, California 90089-2564 USA
E-mail: {sauhsuaw, ubli}@usc.edu, cckuo@sipi.usc.edu

Abstract — **Performance of the output signal-to-interference plus noise ratios (SINRs) and bit error rates (BERs) of the maximum ratio combining (MRC) multistage minimum-mean squared error (MMSE), minimum output energy (MOE), best linear unbiased estimator (BLUE) and maximum likelihood (ML) receivers are analyzed for direct-sequence code-division multiple access (DS-CDMA) systems in multipath Rayleigh fading channels, based on the common multistage structure proposed in [1–3]. The SINRs of these receivers are proved to be monotonically increasing with the number of applied stages. The upper bound of output SINRs and its achievability conditions are also provided, which are shown to form the condition for the occurrence of BER floors of the multistage MRC MMSE/MOE/BLUE/ML multiuser receivers. A rule for selecting the number of stages is thus provided to take full advantage of the multistage structure of these receivers.**

I. Introduction

Reduced-rank linear filtering based on the multistage Wiener filtering (MSWF) [4] for interference suppression of direct-sequence code-division multiple-access (DS-CDMA) systems has been studied for additive white Gaussian noise (AWGN) channels in [5]. A random-code analysis for the steady-state output SINR of the reduced-rank MSWF was conducted in [6] for AWGN channels. An important feature of the MSWF is its achievement of performance comparable to the full-rank output signal-to-interference plus noise ratio (SINR) with relative few stages. Based on the concept of MSWF, a class of linear multistage (MS) receivers based on minimum mean square error (MMSE), minimum output energy (MOE), best linear unbiased estimation (BLUE) estimators/dectors and the maximum likelihood (ML) detector were developed in [1–3] for multiuser detection in multipath Rayleigh fading channels.

Important questions to answer regarding multistage multiuser detectors are whether the SINRs increase with the number of applied stages and whether the SINRs are performance-limited due to multiple access interference (MAI). If the answer to the latter is affirmative, can we established the conditions for avoiding the MAI limiting performance? To analyze these questions, we will first establish the monotonically increasing properties of SINRs of maximum ratio combining (MRC) MMSE/MOE/BLUE/ML filter banks (FBS). We develop upper bounds for the output SINRs; the condition for achieving these bounds will also be given. Using these results and the relationship between SINRs and bit error rates (BERs), the analysis of BER floors of the aforementioned receivers will be conducted for DS-CDMA systems in multipath Rayleigh fading channels. Finally, a simple condition for checking the existence of BER floors and a rule for selecting the number of applied stages for these multistage receivers is developed.

II. System Model

A standard model for asynchronous DS-CDMA modulated with binary phase shift keying (BPSK) is considered in this paper. Assume the maximum path delay for each user is less than one symbol interval T_s. After pulse matched filtering and chip rate sampling, the discrete-time received signal vector $\mathbf{y}$ obtained by collecting N consecutive samples, (N is the spreading gain), is given by [7]

$$\mathbf{y}(m) = \sum_{k=1}^{K} [\mathbf{S_{k+}}\mathbf{A_k}\Gamma_k(m)b_k(m) + \mathbf{S_{k-}}\mathbf{A_k}\Gamma_k(m)b_k(m-1)] + \mathbf{n}(m), \quad (1)$$

where K is the number of users, $\mathbf{A}_k = diag([A_{k1}, \dots, A_{kL_k}])$, $\mathbf{S_{k+}} = [\mathbf{s}_{k1}^{+}, \dots, \mathbf{s}_{kL_k}^{+}]$, $\mathbf{S_{k-}} = [\mathbf{s}_{k1}^{-}, \dots, \mathbf{s}_{kL_k}^{-}]$ and $\Gamma_k = [\gamma_{k1}, \dots, \gamma_{kL_k}]^T$. The number of paths of user k is L_k, A_{kl} is the amplitude of the signal on path l of user k, $\mathbf{s}_{kl}^{+}$ and $\mathbf{s}_{kl}^{-}$ are the partial spreading codes corresponding to the current bit $b(m)$ and the previous bit $b(m-1)$ over the sampling interval, respectively. The filtered noise vector $\mathbf{n}(m)$ is complex Gaussian distributed with zero mean and covariance matrix, $N_0\mathbf{I}$. Finally, the discrete-time fading process $\gamma_{kl}(m)$ over the sampling interval T_s is a complex zero-mean Gaussian process that satisfies $E\{\gamma_{k_1 l_1}^{*}(m)\gamma_{k_2 l_2}(m)\} = \delta(k_1 - k_2)\delta(l_1 - l_2)$, with the autocorrelation of two adjacent samples defined as $\rho = E[\gamma_{k_1 l_1}^{*}(m)\gamma_{k_1 l_1}(m-1)]$. The received signal can also be expressed as,

$$\mathbf{y}(m) = \mathbf{S}_{1+}\mathbf{A}_1\Gamma_1(m)b_1(m) + \mathbf{I}_1(m),$$

$$\text{where } \mathbf{I}_1(m) \equiv \sum_{k=2}^{K}[\mathbf{S}_{k+}\mathbf{A}_k\Gamma_k(m)b_k(m) + \sum_{k=1}^{K}\mathbf{S}_{k-}\mathbf{A}_k\Gamma_k(m)b_k(m-1)] + \mathbf{n}(m) \quad (2)$$

The interference vector $\mathbf{I}_1$ is the aggregate of MAI, inter symbol interference (ISI) and the complex Gaussian noise, with auto-covariance matrix defined as $\mathbf{C}_1 \equiv E(\mathbf{I}_1\mathbf{I}_1^H)$. The vector $\mathbf{I}_1$ is also complex Gaussian distributed because $\mathbf{S}_{k+}$ and $\mathbf{S}_{k-}$ are deterministic and $\mathbf{\Gamma}_k$ is a complex Gaussian vector. For simplicity, we use L to denote L_1 in the remainder of this paper, which is the number of the resolvable paths of the desired user's channel.

III. Common Structure for Interference Suppression

It has been shown in [1–3] that the MS MMSE/MOE/BLUE/ML FBs share the same multistage structure for interference suppression modulo distinctive output scaling matrices as seen in Fig. 1. The method for forming the multistage filters and the corresponding matrices $\mathbf{H}_i^{N\times L}$ and $\mathbf{B}_i = \mathbf{I} - \mathbf{H}_i\mathbf{H}_i^H$, $i = 1, 2, \cdots, D$, of D-stage implementations can

0-7803-7799-0/03/$17.00 © 2003 IEEE

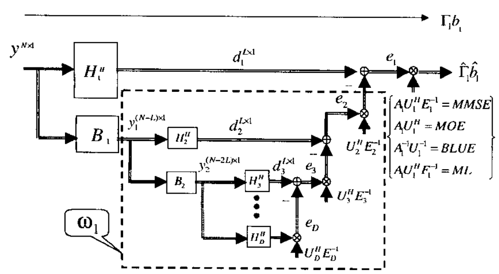

Fig. 1. The common structure for the D-stage MMSE-based multiuser receivers in multi-path fading channels. The output scaling matrix is $\mathbf{A}_1\mathbf{U}_1^H\mathbf{E}_1^{-1}$ for MS-MMSE-FB, $\mathbf{A}_1\mathbf{U}_1^H$ for MS-MOE-FB, $\mathbf{A}_1^{-1}\mathbf{U}_1^{-1}$ for MS-BLUE-FB and $\mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_1^{-1}$ for MS-ML-FB. The objective of estimation is $\Gamma_1 b_1$.

be found in [1]. The filters' soft outputs can be represented with a common form:

$$\begin{aligned} \mathbf{z}_i &\equiv \omega_i \mathbf{T}\mathbf{y} = \mathbf{C}_i[\mathbf{d}_1 - \mathbf{r}_{\mathbf{y}_1\mathbf{d}_1}^H \mathbf{R}_{\mathbf{y}_1}^{-1}\mathbf{y}_1] \\ i &\in mmse, moe, blue, ml, \end{aligned} \quad (3)$$

where $\mathbf{d}_1 \equiv \mathbf{H}_1^H\mathbf{y}$ and $\mathbf{y}_1 \equiv \mathbf{B}_1\mathbf{y}$, $\mathbf{r}_{\mathbf{y}_1\mathbf{d}_1} \equiv E(\mathbf{y}_1\mathbf{d}_1^H)$ is the cross-covariance matrix of $\mathbf{y}_1$ and $\mathbf{d}_1$. The matrices $\mathbf{R}_{\mathbf{d}_1}$ and $\mathbf{R}_{\mathbf{y}_1}$ are the covariance matrices of $\mathbf{d}_1$ and $\mathbf{y}_1$, respectively. The scaling matrices are given by: $\mathbf{C}_{mmse} \equiv \mathbf{A}_1\mathbf{U}_1^H\mathbf{E}_1^{-1}$, $\mathbf{C}_{moe} \equiv \mathbf{A}_1\mathbf{U}_1^H$, $\mathbf{C}_{blue} \equiv \mathbf{A}_1^{-1}\mathbf{U}_1^{-1}$ and $\mathbf{C}_{mmse} \equiv \mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_1^{-1}$, where $\mathbf{U}_1 \equiv \mathbf{H}_1^H\mathbf{S}_{1+}$, is obtained by applying the Gram-Schmidt process to the column vectors of $\mathbf{S}_{1+}$, and $\mathbf{E}_1 \equiv (\mathbf{R}_{\mathbf{d}_1} - \mathbf{r}_{\mathbf{y}_1\mathbf{d}_1}^H\mathbf{R}_{\mathbf{y}_1}^{-1}\mathbf{r}_{\mathbf{y}_1\mathbf{d}_1})$.

The matrix $\mathbf{F}_1 = E[(\mathbf{H}_1^H\mathbf{I}_1 - \omega_1^H\mathbf{B}_1\mathbf{I}_1)(\mathbf{H}_1^H\mathbf{I} - \omega_1^H\mathbf{B}_1\mathbf{I}_1)^H]$, where $\omega_1 \equiv \mathbf{R}_{\mathbf{I}_{B1}}^{-1}\mathbf{r}_{\mathbf{I}_{B1}\mathbf{I}_{H1}} = \mathbf{R}_{\mathbf{y}_1}^{-1}\mathbf{r}_{\mathbf{y}_1\mathbf{d}_1}$ is an MMSE filter bank as shown in Fig. 1. Physical, $\mathbf{F}_1$ is the autocorrelation matrix of the residual interference in the signal space, $\mathcal{H}_1$, filtered by ω_1. The signal subspace, $\mathcal{H}_1$, is an orthonormal space spanning the column space of matrix $\mathbf{H}_1$, and $\mathcal{B}_1$ is the orthonormal complement of $\mathcal{H}_1$, spanning the row space of an appropriated selected $\mathbf{B}_1$.

The matrix $\mathbf{E}_1 = E(\mathbf{e}_1\mathbf{e}_1^H)$, [*cf.* Fig. 1], is the error correlation matrix, where

$$\mathbf{E}_1 = \mathbf{F}_1 + \mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H, \quad (4)$$

and $\mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H$ is the autocorrelation matrix of the projection of $\mathbf{S}_{1+}\mathbf{A}_1$ onto the signal subspace $\mathcal{H}_1$. The multistage structure in Fig. 1 is essentially a multistage interference suppressor using filter ω_1.

IV. Performance for Output SINRs

In this paper we analyze the performance of the multistage MMSE/MOE/BLUE/ML receivers in multipath Rayleigh fading channels. The analysis relies on two key steps: separating the multistage implementations from the desired user's channel coefficients, and constructing the subspace spanned by the multistage implementations of these filter banks. By using the common structure for multistage implementations and employing maximal ratio combining (MRC) for the outputs of the filter banks, where the decision statistic is given by $\varphi(m) = Re(\Gamma^H(m)\mathbf{z}(m))$ [*cf.* (3)], the dependence of performance on the number of applied stages and the average of performance over all possible channel realization can be analyzed separately. The subspace, $\mathcal{T}_{LD}$, spanned by the multistage MRC MMSE/MOE/BLUE/ML FBs is given by:

If the blocking matrix $\mathbf{B}_i$ at each stage i as shown in Fig. 1 is taken to be $\mathbf{B}_i \equiv \mathbf{I} - \mathbf{H}_i\mathbf{H}_i^H$,

$$\mathcal{T}_{LD} \; span \; \mathbf{T}_{LD}^H \equiv [\mathbf{H}_1 | \, \mathbf{H}_2 | \, \ldots \, | \, \mathbf{H}_D] \equiv [\mathbf{H}_1 | \, \mathbf{H}_{1\perp}]. \quad (5)$$

Details of the proof are in [1]. To indicate the number of applied stages, the $\mathbf{E}_1$ term [*cf.* (4)] corresponding to the D-stage transformation using $\mathbf{T}_{LD}$ is denoted as $\mathbf{S}_D$, and can be expressed as, [1]

$$\mathbf{S}_D \equiv \mathbf{H}_1^H\mathbf{R}\mathbf{H}_1 - \mathbf{H}_1^H\mathbf{R}\mathbf{H}_{1\perp}(\mathbf{H}_{1\perp}^H\mathbf{R}\mathbf{H}_{1\perp})^{-1}\mathbf{H}_{1\perp}^H\mathbf{R}\mathbf{H}_1, \quad (6)$$

where the steady-state $\mathbf{R} \equiv E(\mathbf{y}\mathbf{y}^H)$ is given by

$$\mathbf{R} = \sum_{k=1}^{K} \mathbf{S}_{k+}\mathbf{A}_k^2\mathbf{S}_{k+}^H + \mathbf{S}_{k-}\mathbf{A}_k^2\mathbf{S}_{k-}^H + N_0\mathbf{I}. \quad (7)$$

The $\mathbf{F}_1$ term [*cf.* (4)] corresponding to the D-stage implementation is given by $\mathbf{F}_D \equiv \mathbf{S}_D - \mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H$.

Now, we give the sufficient condition for the equality of the output SINRs of the MS MMSE/MOE/BLUE/ML receivers in multipath Rayleigh fading channels.

Proposition 1:
For the D-stage MRC MMSE/MOE/BLUE/ML-FBs, if $\mathbf{A}_1\mathbf{U}_1^H\mathbf{U}_1\mathbf{A}_1 = \kappa_1\mathbf{I}$, $\kappa_1 \geq 0$ and $\mathbf{S}_D = \kappa_2\mathbf{I}$, $\kappa_2 \geq \kappa_1$, then

$$L \geq SINR_{mmse} = SINR_{moe} = SINR_{blue} = SINR_{ml}, \quad (8)$$

where the output SINRs are quoted from [1–3] as follows.

$$SINR_{mmse} = tr(\mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D^{-1}\mathbf{U}_1\mathbf{A}_1) \quad (9)$$

$$SINR_{moe} = \frac{tr^2(\mathbf{A}_1\mathbf{U}_1^H\mathbf{U}_1\mathbf{A}_1)}{tr(\mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D\mathbf{U}_1\mathbf{A}_1)} \quad (10)$$

$$SINR_{blue} = \frac{L^2}{tr((\mathbf{U}_1\mathbf{A}_1)^{-1}\mathbf{S}_D(\mathbf{U}_1\mathbf{A}_1)^{-H})} \quad (11)$$

$$SINR_{ml} = \frac{tr^2[\mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_D^{-1}\mathbf{U}_1\mathbf{A}_1]}{tr[\mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_D^{-1}\mathbf{U}_1\mathbf{A}_1 + (\mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_D^{-1}\mathbf{U}_1\mathbf{A}_1)^2]}. \quad (12)$$

For multipath case, it is unlikely that $\mathbf{S}_D$ (6) and $\mathbf{A}_1\mathbf{U}_1^H\mathbf{U}_1\mathbf{A}_1$ are diagonal matrices. Note that $\mathbf{S}_{1+}$ is a function of the spreading sequence and the path delays of the desired user. Having $\mathbf{A}_1\mathbf{U}_1^H\mathbf{U}_1\mathbf{A}_1 = \mathbf{A}_1\mathbf{S}_{1+}^H\mathbf{S}_{1+}\mathbf{A}_1 = \kappa_1\mathbf{I}$ implies that the column vectors of $\mathbf{S}_{1+}$ are orthogonal to each others, since $\mathbf{A}_1$ is a diagonal matrix. Thus the equality is unlikely to hold in practice. However, Corollary 1 gives a situation where the equality holds naturally.

Corollary 1:
For the multistage MRC MMSE/MOE/BLUE/ML receivers in flat Rayleigh fading channels

$$SINR_{mmse} = SINR_{moe} = SINR_{blue} = SINR_{ml}. \quad (13)$$

Proof:
In flat fading channels, $\mathbf{A}_1, \mathbf{U}_1$ and $\mathbf{S}_D$ are scalars ($L = 1$). Thus, by (4), $\mathbf{S}_D = \kappa_2 \geq \mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H = \kappa_1$. According to Proposition 1, the equality holds for any spreading sequence $\mathbf{S}_{1+}$ of the desired user. ■

From (10) to (12), it is clear that the output SINRs are all related to the kernels of $K_{mmse} \equiv \mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D^{-1}\mathbf{U}_1\mathbf{A}_1$ and $K_{ml} \equiv \mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_D^{-1}\mathbf{U}_1\mathbf{A}_1$. In Section VI, these kernels will also play important roles in the BER analysis of the aforementioned filter banks. To investigate the properties of output SINRs and BERs with respect to the signal power, it is necessary to show that the eigenvalues of this kernels are monotonically increasing with the desired user's power.

Proposition 2: Ignore the self induced ISI, *i.e.* assume $\mathbf{S}_{1-} = \mathbf{0}$. Given that $\alpha_1 > \alpha_2$ and define $\lambda_j^\beta(i)$, $j = 1, \cdots, L$, being the eigenvalues of K_β, $\beta = \{mmse, ml\}$, with amplitude matrix $\alpha_i\mathbf{A}_1$, $i = 1, 2$.
Then $\lambda_j^\beta(1) > \lambda_j^\beta(2)$, $\forall j$, $\forall \beta$.

V. Convergence Analysis of Output SINRs

For multistage implementations, a key question is whether additional stages yield an SINR improvement and if so, can the improvement be quantified? For ML filterbanks, the average SINR is computed as the trace of $K_{ml} = \mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_D^{-1}\mathbf{U}_1\mathbf{A}_1 \equiv \mathbf{Q}\Lambda_F\mathbf{Q}^H$ [3]. For the other three filterbanks, the evaluation of the average SINRs is intractable. However, the output SINRs can be determined [3] from (9) to (11) where it is clear that the output SINRs are functions of $K_{mmse} = \mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D^{-1}\mathbf{U}_1\mathbf{A}_1 \equiv \mathbf{Q}\Lambda\mathbf{Q}^H$. Thus answering the question of interest boils down to characterizing the properties of Λ and Λ_F, which requires the exploration of the structure of $\mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D^{-1}\mathbf{U}_1\mathbf{A}_1$ as a function of the number of applied stages.

Given that $\mathbf{A}_1$ and $\mathbf{S}_{1+}$ are fixed, $\mathbf{U}_1$ is a constant matrix. The evolution of the output SINRs, as a function of the number of stages, depends solely on the evolving structure of $\mathbf{S}_D^{-1}$ given by

$$\begin{aligned}\mathbf{S}_D^{-1} &= [\mathbf{R}_{d_1} - \mathbf{U}_2^H[\mathbf{R}_{d_2} - \mathbf{U}_3^H[\mathbf{R}_{d_3}\cdots\mathbf{U}_{D-1}^H[\mathbf{R}_{d_{D-1}} \\ &- \mathbf{U}_D^H\mathbf{R}_{d_D}^{-1}\mathbf{U}_D]^{-1}\mathbf{U}_{D-1}]^{-1}\cdots]^{-1}\mathbf{U}_3]^{-1}\mathbf{U}_2]^{-1} \quad (14)\end{aligned}$$

where $E(\mathbf{y}_{i-1}\mathbf{d}_{i-1}^H) = \mathbf{H}_i\mathbf{U}_i$, $i \in \{2,\cdots,D\}$, can be obtained by a QR factorization. Recall the structure of the MSWF [2,4]. We prune the filter bank at stage i, $i < D$, the pruned portion from stage $i+1$ to stage D forms a reduced-rank MMSE filter bank ω_i which minimizes $E\|\mathbf{d}_i - \omega_i^H\mathbf{y}_i\|^2$. The corresponding error correlation matrix $\mathbf{E}_i = E[(\mathbf{d}_i - \omega_i^H\mathbf{y}_i)(\mathbf{d}_i - \omega_i^H\mathbf{y}_i)^H]$ is equal to $(\mathbf{R}_{\mathbf{d}_i} - \mathbf{r}_{\mathbf{y}_i\mathbf{d}_i}^H\mathbf{R}_{\mathbf{y}_i}^{-1}\mathbf{r}_{\mathbf{y}_i\mathbf{d}_i})$, with $\mathbf{E}_D = \mathbf{R}_{\mathbf{d}_D}$ for the last stage and $SINR_{MMSE} = tr(\mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D^{-1}\mathbf{U}_1\mathbf{A}_1)$, (*cf.* (9), where $\mathbf{S}_D$ replaces $\mathbf{E}_1$ for the D-stage implementation). Therefore, the output SINR of filter ω_i, denoted as $SINR_{mmse}^i$, is equal to $tr(\mathbf{U}_i^H\mathbf{E}_i^{-1}\mathbf{U}_i)$, $2 \le i \le D-1$. [1] Similar to (14) for $\mathbf{S}_D$, $\mathbf{E}_i = [\mathbf{R}_{d_i} - \mathbf{U}_{i+1}^H[\cdots[\mathbf{R}_{d_{D-1}} - \mathbf{U}_D^H\mathbf{R}_{d_D}^{-1}\mathbf{U}_D]\cdots]\mathbf{U}_{i+1}]$. Therefore, $\mathbf{E}_i$ possesses a recursive structure:

$$\mathbf{E}_i = [\mathbf{R}_{d_i} - \mathbf{U}_{i+1}^H\mathbf{E}_{i+1}^{-1}\mathbf{U}_{i+1}],\ i = 1,\cdots,D-1. \quad (15)$$

with $\mathbf{E}_D = \mathbf{R}_{d_D}$. The implication of this structure can be clearly seen for flat Rayleigh fading channels.

Let α_i denote $\mathbf{U}_i^H\mathbf{U}_i$ and β_i denote $\mathbf{R}_{d_i}$ for the scalars resulting from the flat fading case. The output SINR of the matched-filter $\mathbf{H}_i$, denoted by $SINR_{mf}^i$, at each stage i can been shown equal to $\frac{\alpha_i}{\beta_i}$. The output SINR of the MMSE filter ω_i constructed with stage i to stage D is equal to

$$SINR_{mmse}^i = \frac{\alpha_i}{\beta_i - SINR_{mmse}^{i+1}} \quad (16)$$

where, for the last stage, $SINR_{mmse}^D$ is simply equal to $SINR_{mf}^D = \frac{\alpha_D}{\beta_D}$. We observe that the improvement in the MMSE output SINR over that of the matched-filter is given by the subtraction of $SINR_{mmse}^{i+1}$ from the denominator of $SINR_{mf}^i$. As a result, the output SINR of the multistage MMSE receiver over a flat Rayleigh fading channel can be expressed as a continued-fraction of the form

$$SINR_{mmse} = \cfrac{\alpha_1}{\beta_1 - \cfrac{\alpha_2}{\beta_2 - \cfrac{\alpha_3}{\ddots \, \beta_D}}} = \frac{\alpha_1}{\mathbf{S}_D}, \quad (17)$$

where $\alpha_1 = \mathbf{A}_1\mathbf{U}_1^H\mathbf{U}_1\mathbf{A}_1$. Note that α_1 and $\mathbf{S}_D$ are both scalars in this case.

[1] For the convenience of analysis, we have made $\mathbf{A}_1$ explicit in expressing the the cross-correlation $E(\mathbf{y}\mathbf{d}^H) = \mathbf{H}_1\mathbf{U}_1\mathbf{A}_1$ of the first stage, where $\mathbf{d} \equiv \Gamma_1 b_1$. For stage 2 to stage D, $E(\mathbf{y}_{i-1}\mathbf{d}_{i-1}^H) = \mathbf{H}_i\mathbf{U}_i$, $i \in \{2,\cdots,D\}$.

For scalar cases, it is clear from (17) that the output SINR increases with additional stages. For multipath fading channels, we need to show that the eigenvalues of Λ increase with the number of stages. For the convenience of analysis, we define $\mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D^{-1}\mathbf{U}_1\mathbf{A}_1 \equiv \mathbf{Q}_{1:D}\Lambda_{1:D}\mathbf{Q}_{1:D}^H$ and $\mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_{D-1}^{-1}\mathbf{U}_1\mathbf{A}_1 \equiv \mathbf{Q}_{1:(D-1)}\Lambda_{1:(D-1)}\mathbf{Q}_{1:(D-1)}^H$ to be the kernels, K_{mmse}, of the D-stage and (D-1)-stage MMSE filter banks, respectively. The eigendecompositions of the two cases are related as follows:

Theorem 1: $\Lambda_{1:D} > \Lambda_{1:(D-1)}$, $2 \le D \le N$.

For ML filter banks, define $\mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_D^{-1}\mathbf{U}_1\mathbf{A}_1 \equiv \mathbf{Q}_{1:D}\Lambda_{F_{1:D}}\mathbf{Q}_{1:D}^H$ and $\mathbf{A}_1\mathbf{U}_1^H\mathbf{F}_{D-1}^{-1}\mathbf{U}_1\mathbf{A}_1 \equiv \mathbf{Q}_{1:(D-1)}\Lambda_{F_{1:(D-1)}}\mathbf{Q}_{1:(D-1)}^H$ to be the kernels, K_{ml}, of the D-stage and (D-1)-stage ML filter banks, respectively. It can be shown that $\mathbf{I} = (\mathbf{I}+\Lambda_F)(\mathbf{I}-\Lambda)$, this fact can be used to prove:

Theorem 2: $\Lambda_{F_{1:D}} > \Lambda_{F_{1:(D-1)}}$, $2 \le D \le N$.

These two theorems show that the output SINRs for MMSE/MOE/BLUE filterbanks and the average SINR for ML filterbank increase with the number of applied stages.

VI. LIMITING PERFORMANCE

To analyze the limiting performance of BERs, we evaluate the relationship between BER and the SINR. Taking the MRC-MMSE and ML receivers as examples, the moment generating functions of their associated decision statistics can be shown to be, [3]

$$\begin{aligned}M_{mmse}(s) &= s\cdot det[\mathbf{I} - \frac{s}{2}(\Lambda^{\frac{1}{2}} - \Lambda)]det[\mathbf{I} + \frac{s}{2}(\Lambda^{\frac{1}{2}} + \Lambda)] \quad (18)\\ M_{ml}(s) &= s\cdot det[\mathbf{I} - \frac{s}{2}((\Lambda_F + \Lambda_F^2)^{\frac{1}{2}} - \Lambda_F)] \\ & \quad det[\mathbf{I} + \frac{s}{2}((\Lambda_F + \Lambda_F^2)^{\frac{1}{2}} + \Lambda_F)], \quad (19)\end{aligned}$$

the corresponding BERs are determined by evaluating the residual of $M^{-1}(s)$ with respect to its right half plane poles. It is clear from the above formulae that BERs approach zero if $\Lambda = \mathbf{I}$ or $\Lambda_F \sim \infty$. Since $\mathbf{I} = (\mathbf{I}+\Lambda_F)(\mathbf{I}-\Lambda)$ and $\Lambda \le \mathbf{I}$, $\Lambda_F \sim \infty$ if $\Lambda = \mathbf{I}$. It suffices to check the achievability condition on the upper bound of Λ. If Λ saturates at a value smaller than $\mathbf{I}$, there will be BER floors.

Proposition 3: $\Lambda = \mathbf{I}$ if and only if

$$\begin{aligned}\mathbf{F}_D &= E[(\mathbf{H}_1^H\mathbf{I}_1 - \omega_1^H\mathbf{H}_{1\perp}^H\mathbf{I}_1)(\mathbf{H}_1^H\mathbf{I}_1 - \omega_1^H\mathbf{H}_{1\perp}^H\mathbf{I}_1)^H] \\ &= (\mathbf{H}_1^H - \omega_1^H\mathbf{H}_{1\perp}^H)\mathbf{C}_1(\mathbf{H}_1 - \mathbf{H}_{1\perp}\omega_1) = \mathbf{0}. \quad (20)\end{aligned}$$

Recall that $\mathbf{S}_D = \mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H + \mathbf{F}_D$ and $\mathbf{A}_1\mathbf{U}_1^H\mathbf{S}_D^{-1}\mathbf{U}_1\mathbf{A}_1 \equiv \mathbf{Q}\Lambda\mathbf{Q}^H$. Therefore, $\Lambda = \mathbf{I}$ if and only if $\mathbf{S}_D = \mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H$.

The conditions of this proposition to ensure that$\Lambda = \mathbf{I}$ cannot be achieved in practice. Since the dimension of ω_1 is $(D-1)L \times L$ and the rank of $\mathbf{C}_1$ is N an ω_1 does not exist such that $\mathbf{F}_D = \mathbf{0}$. However, in high SNR,*i.e.* $N_0 = 0$, we see that the desired condition can be achieved.

For the analysis at high SNR, we define the rank of $\mathbf{C}_1$ with $N_0 = 0$ as

$$D_{int} = \text{rank}(E[(\mathbf{I}_1 - \mathbf{n})(\mathbf{I}_1 - \mathbf{n})^H]), \quad (21)$$

Achieving $\Lambda \sim \mathbf{I}$ depends on whether a filter can suppress MAI and ISI efficiently. For linear receivers, intuitively, this property is related to the available degrees of freedom, D_{free}, for suppressing interference. Recall the structure of multistage filtering in Fig. 1. The first stage is a filter matched to the desired user's signal in no interference. The stages remaining, from the second to the D*th* stage are dedicated to suppressing interference. We denote $D_{free} = (D-1)\times L$ as the dimensions available for suppressing interference. The achievability condition for the high SNR regime is given by

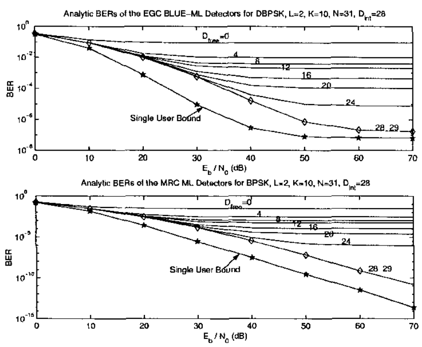

Fig. 2. BERs of the reduced-rank EGC BLUE-ML-FB vs. MRC-ML-FB. $L = 2,\ K = 10,\ N = 31$. The top one is for EGC BLUE-ML-FB and the lower is for MRC-ML-FB.

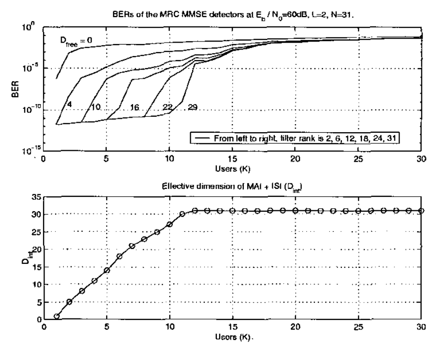

Fig. 3. BERs of the reduced-rank MRC-MMSE-FB. $L = 2,\ N = 31$. From left to right, $D = 1, 3, 6, 9, 12$ and the full-rank.

Proposition 4: $\Lambda = \mathbf{I}$ if $D_{free} \geq D_{int}$ and $N_0 = 0$.

Proof: Define $\mathbf{C}_I \equiv E[(\mathbf{I}_1 - \mathbf{n})(\mathbf{I}_1 - \mathbf{n})^H]$, then $\mathbf{C}_1 = E[\mathbf{I}_1\mathbf{I}_1^H] = \mathbf{C}_I + N_0\mathbf{I}$. As $N_0 = 0$, $\mathbf{S}_D$ can be manipulated into

$$\mathbf{S}_D = \mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H + (\mathbf{H}_1^H - \omega_1^H\mathbf{H}_{1\perp}^H)\mathbf{C}_I(\mathbf{H}_1 - \mathbf{H}_{1\perp}\omega_1). \quad (22)$$

Let $\mathbf{C}_I = \mathbf{Q}_I\Sigma\mathbf{Q}_I^H$, $\Sigma = diag(\sigma_1, \cdots, \sigma_{D_{int}})$, $0 \leq D_{int} \leq N$. $F_D = (\mathbf{H}_1^H - \omega_1^H\mathbf{H}_{1\perp}^H)\mathbf{Q}_I\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}\mathbf{Q}_I^H(\mathbf{H}_1 - \mathbf{H}_{1\perp}\omega_1)$. In order for $\mathbf{S}_D = \mathbf{U}_1\mathbf{A}_1^2\mathbf{U}_1^H$ we require that $\mathbf{H}_1^H\mathbf{Q}_I\Sigma^{\frac{1}{2}} - \omega_1^H\mathbf{H}_{1\perp}^H\mathbf{Q}_I\Sigma^{\frac{1}{2}} = \mathbf{0}$. For a D-stage implementation, $\mathbf{H}_{1\perp}^H$ is of dimension $N \times (D-1)L$ and ω_1 is of dimension $(D-1)L \times L$, $D_{free} = (D-1)L$. Thus, $\mathbf{H}_1^H\mathbf{Q}_I\Sigma^{\frac{1}{2}}$ is of $L \times D_{int}$ and $\mathbf{H}_{1\perp}^H\mathbf{Q}_I\Sigma^{\frac{1}{2}} : D_{free} \times D_{int}$. For each column of ω_1, there are D_{int} equations for D_{free} variables. Thus we require $D_{free} \geq D_{int}$ to solve for ω_1. ■

While Proposition 2 shows that the individual eigenvalues of Λ increase with SNR, Proposition 4 shows that BER floors will be incurred if $D_{free} < D_{int}$ as the resulting $\Lambda < \mathbf{I}$ independent of input SNR. These results show that even for the idealized case of maximal ratio combining (the channel state is implicitly known), BER floors can occur.

Fig. 2 illustrates this limiting performance with simulation results for $K = 10$, $L = 2$ and $D_{int} = 28$. As we can see, when $D_{free} \geq D_{int}$, for D=15 and for full-rank, BER floors can be avoided for MRC schemes and performance is very close to the single user bound. We note that for equal gain combining, error floors always exist. This fact is analyzed in [3].

Fig. 3 shows the effect of achieving $\Lambda = \mathbf{I}$ on the BER at high SNR. In this simulation study, the SNR, E_b/N_0 is set to $60dB$ such that the noise is essentially negligible. Since L=2 for each user, $D_{free} = 2(D-1)$ where D is the number of applied stages. The lower plot shows the effective interference dimension, D_{int}, with respect to the number of users in the system. Cross-referencing to the upper plot, we can see that BERs jump up extremely fast whenever the condition $D_{free} - D_{int} \geq 0$ is violated. These simulation results echo the conclusion of Proposition 4, as long as a reduced-rank filter can sustain the requirement of $D_{free} \geq D_{int}$, interference can be effectively suppressed regardless of filters' ranks in the high SNR regime.

In scenarios where the strength of Gaussian noise is comparable to that of MAI plus ISI, or whenever a system is highly loaded such that $D_{free} < D_{int}$, Λ will quickly saturate with the number of stages, yielding the predicted BER floor for MRC schemes. Thus, to fully take the advantage of reduced-rank filtering, a simple rule for choosing the number of stages is suggested: If the system is not fully loaded, *i.e.* $D_{int} \leq (N-1) \times L$, then choose D as low as possible to exceed the threshold $(D-1) \times L \geq D_{int}$. Otherwise, set $D = 6$ where Λ usually has saturated based on simulation results. The same number of 6 was also observed in [6] according to the results of large-system analysis.

VII. Conclusions

In this paper, we provide performance analysis for a class of multistage filters based on several estimation strategies. The output/average SINR is computed as is the associated probability of error. It is shown that the output/average SINR monotonically increases as the number of applied stages increases. We also show that even for maximal ratio combining an error floor can exist if a key condition relating to interference suppression is not met. A simple design rule is provided for the number of stages necessary for good performance. We have been able to extend our analysis of maximal ratio combining schemes to that of equal gain combining as well [3].

Acknowledgments

This research has been funded in part by the Integrated Media Systems Center, a National Science Foundation Engineering Research Center, Cooperative Agreement No. EEC-9529152.

References

[1] S.-H. Wu, U. Mitra, and C.-C. Kuo, "Multi-stage MMSE/MOE receivers for frequency-selective fading channels in DS-CDMA systems," in *Proc. IEEE Globecom.* Taipei, Taiwan, Nov 2002.

[2] S.-H. Wu, U. Mitra, and C.-C. Kuo, "Perfprmance analysis of multistage BLUE/MMSE receivers for DS-CDMA in frequency-selective fading channels," in *Proc. IEEE ICC.* Alaska, May 2003, to be presented.

[3] S.-H. Wu, U. Mitra, and C.-C. Kuo, "A common framework for blind multistage multiuser receivers of DS-CDMA in frequency-selective fading channels," submitted to Globecom 2003.

[4] J. S. Goldstein, I. S. Reed, and L. L. Scharf, "A multistage representation of the wiener filter based on orthogonal projections," *IEEE Trans. on Information Theory*, vol. 44, no. 7, pp. 2943–2959, Nov. 1998.

[5] M. L. Honig and J. S. Goldstein, "Adaptive reduced-rank interference suppression based on the multistage wiener filter," *IEEE Trans. on Communications*, vol. 50, no. 6, pp. 986–994, June 2002.

[6] M. L. Honig and Weimin Xiao, "Performance of reduced-rank linear interference suppression," *IEEE Trans. on Information Theory*, vol. 47, no. 5, pp. 1928–1946, July 2001.

[7] S. L. Miller, M. L. Honig, and L. B. Milstein, "Performance analysis of MMSE receivers for DS-CDMA in frequency-selective fading channels," *IEEE Trans. on Communications*, vol. 48, no. 11, pp. 1919–1929, Nov. 2000.